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Pricing of spread options on stochastically correlated underlyings

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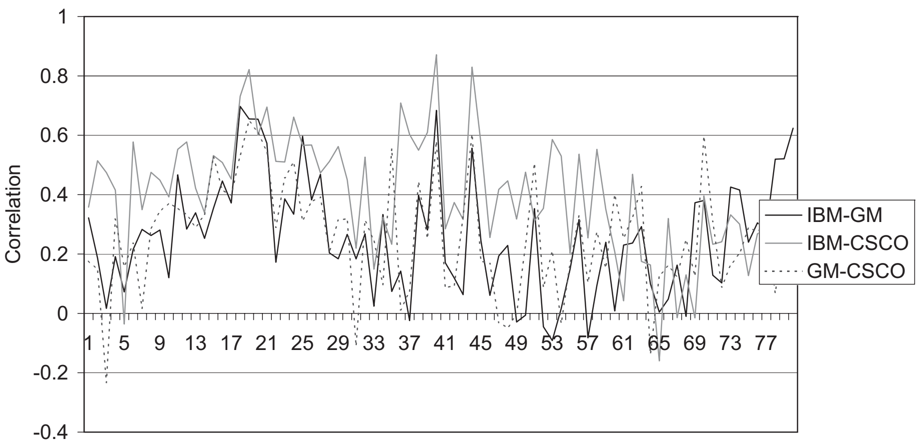
This paper proposes a method to price spread options on stochastically correlated underlying assets. Therefore, it provides a more realistic approach towards a dependence structure. We generalize a constant correlation tree algorithm developed by Hull (2002) and extend it using the notion of stochastic correlation. The resulting tree algorithm is recombining and easy to implement. Moreover, the average price error decreases approximately as $n^{-1.6}$, where n is the number of time steps. We show that this level of convergence is similar to that of the algorithm for the constant correlation case. Our sensitivity analysis with respect to the stochastic correlation parameters shows that the constant correlation model systematically overprices spread options on two stochastically correlated underlying assets. Furthermore, we use our model to derive hedge ratios for the correlation of a spread option and show that the constant correlation model also overprices the hedge ratios.

1 INTRODUCTION

A spread option is a derivative on the difference of two underlying assets with a terminal payoff of the form $[S_1(T) - S_2(T) - K]^+$, where $S_1(T)$ and $S_2(T)$ denote the values of the underlying assets in T and K denotes the exercise price. The main challenge in pricing spread options lies in the lack of knowledge about the distribution of the difference between two correlated stochastic processes (see Dempster and Hong (2000)). Among the different approaches to spread option pricing is the arithmetic Brownian motion model, in which the prices of the underlyings as well as the spread are modeled using Brownian motions with constant correlations (see

FIGURE 1 Structure of the correlations for a window of 50 days.



Poitras (1998)). This setting allows a closed-form solution but does not prevent negative values for the underlyings. Other approaches such as those of Carmona and Durrleman (2003), Pearson (1995) or Shimko (1994) model the underlying assets as geometric Brownian motions assuming constant correlation. In a recent paper, Dempster and Hong (2000) introduced a reasonably fast numerical method to price these derivatives under the framework of stochastic volatility.

In plain vanilla option pricing the assumptions of the Black–Scholes model on volatility have been relaxed by the works of Hull and White (1990, 1996), Stein and Stein (1991), Heston (1993) and Shu and Zhang (2003). However, so far the correlation structure has hardly been addressed even though there are many papers which find evidence for stochastically changing correlations. Among more recent papers, Ramchmand and Susmel (1998) use a switching autoregressive conditional heteroscedasticity (ARCH) technique to find evidence for differences in correlations across variance regimes. Using data from international stock markets Ball and Torous (2000) show that the estimated correlation structure is dynamically changing over time. Before, Makridakis and Wheelwright (1974) found that international correlations are unstable over time and Kaplanis (1988) rejected the null hypothesis of constant correlations comparing the matrices of monthly returns of 10 markets. However, even within a single market correlations seem to change stochastically, which can be seen from the correlations computed for a 50-day time window on the time series of IBM, GM and Cisco stocks from 1986 to 2006 (see Figure 1). The stochastic nature is evident but, as explained before, so far in literature there have been several impulses and suggestions by Dempster and Hong (2000) and Dupire (1993) of how to handle stochastic correlation. In particular Collin-Dufresne and Goldstein (2001) introduce stochastic correlation in a generalized-affine framework to price caps and swaptions.

In this paper we want to relax the assumption of constant correlation found in most of the existing literature concerning spread option pricing. We price spread options on stochastically correlated underlying assets using a bivariate binomial tree model. The tree model generalizes a constant correlation tree model developed by Hull (2002) and extends it using the notion of stochastic correlation. Hull's constant correlation tree model does not impose any restrictions on the correlation structure which eases the introduction of stochastic correlation. The advantage of the Hull method is that the tree is recombining because the increments of the up and down jumps of the singular assets are independent of the correlation structure. Thus, despite the introduction of stochastic correlation, our method is easy to implement and the numerical convergence is fast. This stochastic correlation model allows for a more realistic approach to correlation structures. Our sensitivity analysis with respect to the stochastic correlation parameters shows that the Hull constant correlation model systematically overprices spread options on two stochastically correlated underlying assets. Furthermore, we provide more realistic hedging parameters for the correlation of a spread option priced with our method.

We propose a structure for the underlying processes in Section 2. In Section 3 the bivariate binomial tree model for constant correlation is derived in detail. In Section 4 we describe the numerical approximation of the stochastic correlation using trinomial trees. We combine the numerical approximation of the underlyings and the stochastic correlation in Section 5. In Section 6 we analyze the sensitivity of the price of the spread option with respect to the parameters of the stochastic correlation process and provide the hedging parameters for the spread option. We provide conclusions in Section 7.

2 UNDERLYING PROCESSES

To model the correlation we propose a transformation[1] $y(t)$ of the correlation, which maps its distribution from $[-1; 1]$ to $(-\infty, \infty)$. We found that the real correlation data under this transformation followed a mean reverting process. The system of processes is defined on a filtered probability space $(\Omega, \mathcal{F}, \tilde{\mathcal{Q}}, \mathbb{F})$ where $\mathcal{F}_0$ contains all subsets of the $(\tilde{\mathcal{Q}}-)$ null sets of $\mathcal{F}$ and $\mathbb{F}$ is right-continuous. As we assume that the market is complete the processes are defined under the risk-neutral measure $\tilde{\mathcal{Q}}$. We propose the following system of underlying processes:

$$\mathrm{d}S_i = S_i r \,\mathrm{d}t + S_i \sigma_i \,\mathrm{d}W_i \quad \text{for } i \in \{1, 2\} \tag{1}$$

$$\rho(y) = 1 - 2\exp(-\exp(y_t)) \tag{2}$$

$$\mathrm{d}y = a(b - y_t)\,\mathrm{d}t + c\,\mathrm{d}Z \tag{3}$$

where:

$$E[\mathrm{d}W_1\,\mathrm{d}W_2 | \mathcal{F}_t] = \rho(t)\,\mathrm{d}t \tag{4}$$

$$E[\mathrm{d}W_i\,\mathrm{d}Z] = 0 \tag{5}$$

[1]We applied several transformations to the data. The transformation proposed here provided the best fit to the data in terms of deviation from the model assumptions of Gaussianity for $y(t)$.

Here S_i are the prices of the two stocks, σ_i, r, a, b and c are fixed constants, and $\mathrm{d}W$ and $\mathrm{d}Z$ are Wiener processes that are independent. The correlation is governed by an arithmetic Ornstein–Uhlenbeck process, with a tendency to revert back to a long-run average level of b.

3 BINOMIAL TREE MODEL FOR TWO ASSETS WITH CONSTANT CORRELATION

To construct the model with constant correlation, the assets are assumed to follow a geometric Brownian motion with constant drift and volatility, $\mathrm{d}S_i = S_i \mu_i \,\mathrm{d}t + S_i \sigma_i \,\mathrm{d}W_i,\ i \in \{1, 2\}$ (see (1)).

Therefore, $S_i(t) = S_i(0)\exp\{(r - \frac{1}{2}\sigma_i^2)t + \sigma_i W_i(t)\}$. The constant correlation is defined by $E[\mathrm{d}W_1\,\mathrm{d}W_2 | \mathcal{F}_t] = \rho\,\mathrm{d}t$.

The continuous bivariate lognormal distribution of the prices of the two assets is approximated with a four-jump discrete distribution. This discrete approximation converges to the lognormal distribution as the length of time steps tends to zero.

For the binomial approximation the lifetime of the option is divided into $n = T/\Delta t$ equal time steps, where Δt is the length of one time step. It is assumed that both assets can jump to two different values at each time step: the assets can increase after one time step by u_i (u_j) with probability p_i (p_j) or fall by d_i (d_j) with $1 - p_i$ $(1 - p_j)$, respectively. Thus, if $S_1(t)$ and $S_2(t)$ are the values of the two assets at time step t, then the values of $S_1(t+1)$ and $S_2(t+1)$ can be any of the combinations:

$$\begin{array}{llll} u_1 S_1 & u_2 S_2 & \text{with probability} & p_a \\ u_1 S_1 & d_2 S_2 & \text{with probability} & p_b \\ d_1 S_1 & u_2 S_2 & \text{with probability} & p_c \\ d_1 S_1 & d_2 S_2 & \text{with probability} & p_d \end{array}$$

with:

$$p_a + p_b + p_c + p_d = 1 \tag{6}$$

$$p_a + p_b = p_1 \tag{7}$$

$$p_d + p_c = 1 - p_1 \tag{8}$$

$$p_b + p_d = 1 - p_2 \tag{9}$$

$$p_a + p_c = p_2 \tag{10}$$

The nodes in the tree are denoted by (i, j, t), where i and j denote the number of upwards moves of the first and second asset, respectively, and t is the time $(t\Delta t)$ that has passed since $t = 0$. Thus, in a recombining tree the possible number of combinations of the stock prices after a jump at time t is $(t+1)^2$. This interrelationship between the number of time steps and combinations ensures that the numerical algorithm is not exponentially dependent in time.

We determine the probabilities and the move sizes u_i, d_i by requiring that the four-point discrete distribution converges to the corresponding continuous distribution.

PROPOSITION 1 (Bidimensional binomial approximation) *We choose the parametrization of the previously defined bidimensional binomial model so that the expected values of the discrete model $E(S_i) \to S_i e^{r\Delta t}$ (see (11) and (12)) as well as the variances $Var(S_i) \to \sigma_i^2 \Delta t$ (see (13) and (14)) and the covariance $Cov(S_i, S_j) \to \sigma_i \sigma_j \rho \Delta t$ (see (15)) converge to their continuous counterparts as $\Delta t \to 0$ (see Rubinstein (2000) or Kamrad and Richtken (1991)). These conditions can be slightly simplified:*

$$e^{r\Delta t} = u_1 p_1 + (1 - p_1) d_1 \tag{11}$$

$$e^{r\Delta t} = u_2 p_2 + (1 - p_2) d_2 \tag{12}$$

$$e^{r\Delta t}(u_1 + d_1) - u_1 d_1 - e^{2r\Delta t} = \sigma_1^2 \Delta t \tag{13}$$

$$e^{r\Delta t}(u_2 + d_2) - u_2 d_2 - e^{2r\Delta t} = \sigma_2^2 \Delta t \tag{14}$$

$$\begin{aligned} &u_1 u_2 p_a + u_1 d_2 p_b + d_1 u_2 p_c + d_1 d_2 p_d \\ &\quad - (u_1 p_1 + (1 - p_1) d_1)(u_2 p_2 + (1 - p_2) d_2) = \sigma_1 \sigma_2 \rho \Delta t \end{aligned} \tag{15}$$

which in terms of p_i lead to the following expressions:

$$p_a = p_1 p_2 - \frac{\sigma_1 \sigma_2 \Delta t p_1 p_2 \rho}{(e^{r\Delta t} - d_1)(d_2 - e^{r\Delta t})} \tag{16}$$

$$p_b = p_1(1 - p_2) + \frac{\sigma_1 \sigma_2 \Delta t p_1 p_2 \rho}{(e^{r\Delta t} - d_1)(d_2 - e^{r\Delta t})} \tag{17}$$

$$p_c = p_2(1 - p_1) + \frac{\sigma_1 \sigma_2 \Delta t p_1 p_2 \rho}{(e^{r\Delta t} - d_1)(d_2 - e^{r\Delta t})} \tag{18}$$

$$p_d = (1 - p_1)(1 - p_2) - \frac{\sigma_1 \sigma_2 \Delta t p_1 p_2 \rho}{(e^{r\Delta t} - d_1)(d_2 - e^{r\Delta t})} \tag{19}$$

while ρ might be restricted by the choice of the probabilities and the move sizes as we have to impose $0 \le p_i \le 1$:

$$\frac{(p_1 p_2 - 1)(e^{r\Delta t} - d_1)(d_2 - e^{r\Delta t})}{\sigma_1 \sigma_2 p_1 p_2 \Delta t} \le \rho \le \frac{(e^{r\Delta t} - d_1)(d_2 - e^{r\Delta t})}{\sigma_1 \sigma_2 \Delta t}$$

$$\begin{aligned} &\frac{p_1(p_2 - 1)(e^{r\Delta t} - d_1)(d_2 - e^{r\Delta t})}{\sigma_1 \sigma_2 p_1 p_2 \Delta t} \le \rho \\ &\quad \le \frac{(1 - p_1(1 - p_2))(e^{r\Delta t} - d_1)(d_2 - e^{r\Delta t})}{p_1 p_2 \sigma_1 \sigma_2 \Delta t} \end{aligned}$$

$$\frac{p_2(1-p_1)(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{\sigma_1\sigma_2 p_1 p_2\Delta t} \leq \rho$$
$$\leq \frac{(1-p_2(1-p_1))(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{p_1 p_2\sigma_1\sigma_2\Delta t}$$
$$\frac{((1-p_1)(1-p_2)-1)(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{\sigma_1\sigma_2 p_1 p_2\Delta t} \leq \rho$$
$$\leq \frac{(1-p_1)(1-p_2)(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{p_1 p_2\sigma_1\sigma_2\Delta t}$$

For a proof see Appendix A.

There is some degree of freedom in the choice of the parameters in the tree method as there are fewer conditions as parameters, which allows us either to choose the probabilities p_i or u_i and d_i. If we use parameters such as the Cox–Ross–Rubinstein model[2] the correlation has to be restricted to a proper subset of the interval $[-1, 1]$. In the following, we choose $p_i = 0.5$ as stated in the following corollary.

COROLLARY 1 *Let $p_i = 0.5$ for $i = a, b, c, d$. Then, the correlations are not constrained by the method, ie, the correlation can be any value in the range $-1 \leq \rho \leq 1$.*

PROOF From (11) and (12) we obtain:

$$2e^{r\Delta t} = u_1 + d_1, \quad 2e^{r\Delta t} = u_1 + d_1 \tag{20}$$

Equation (13) reduces to:

$$u_i^2 - 2e^{r\Delta t}u_i + e^{2r\Delta t} - \sigma_i^2\Delta t = 0$$

This is solved by:

$$u_i = e^{r\Delta t} + \sigma_i\sqrt{\Delta t}, \quad d_i = e^{r\Delta t} - \sigma_i\sqrt{\Delta t}, \quad i \in \{1, 2\} \tag{21}$$

[2]The Cox–Ross–Rubinstein one-dimensional tree model specifies $u_i = e^{\sigma_i\sqrt{\Delta t}}$, $d_i = e^{-\sigma_i\sqrt{\Delta t}}$, $p_i = \frac{1}{2} + \frac{1}{2}(r/\sigma_i)\sqrt{\Delta t}$. In this case ρ is restricted by:

$$\begin{aligned}
&p_a : -4p_1p_2 \leq \rho \leq 4(1 - p_1p_2)\\
&p_b : 4(p_1(1-p_2)-1) \leq \rho \leq 4p_1(1-p_2)\\
&p_c : 4(p_2(1-p_1)-1) \leq \rho \leq 4p_2(1-p_1)\\
&p_d : -4(1-p_1)(1-p_2) \leq \rho \leq 4(1-(1-p_1)(1-p_2))
\end{aligned}$$

where:

$$p_i = \frac{1}{2} + \frac{1}{2}\frac{r}{\sigma_i}\sqrt{\Delta t}$$

FIGURE 2 Relation between time steps t and $t+1$ when $\kappa = l$.

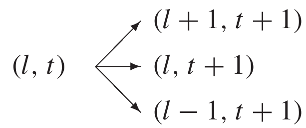


FIGURE 3 Relation between time steps t and $t+1$ when $\kappa = l+1$.

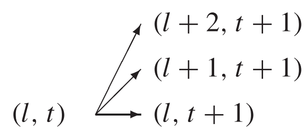
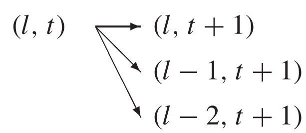


FIGURE 4 Relation between time steps t and $t+1$ when $\kappa = l-1$.



Substituting (21) into (A.17) it can be shown that:

$$p_a = \tfrac{1}{4} + \tfrac{1}{4}\rho, \quad p_b = \tfrac{1}{4} - \tfrac{1}{4}\rho, \quad p_c = \tfrac{1}{4} - \tfrac{1}{4}\rho, \quad p_d = \tfrac{1}{4} + \tfrac{1}{4}\rho \tag{22}$$

and it follows that the probabilities are positive for any ρ, $-1 \leq \rho \leq 1$. □

It should be noted that this paper does not present a formal proof of the convergence of the algorithm to the true price of the derivative under consideration (spread option).

4 NUMERICAL IMPLEMENTATION OF THE MEAN-REVERTING PROCESS

The process (3) is implemented using the trinomial tree suggested by Hull and White (1990). In the following, nodes are denoted by (l, t), where l is the number of upwards movements, ie, the value $y(l, t) = y(0) + l\Delta y$, and t indicates the number of time steps passed since $t = 0$. For the implementation of (3) the three branching methods illustrated in Figures 2–4 are applied, where $\kappa = l$, $\kappa = l + 1$ and $\kappa = l - 1$, respectively.

The probabilities are derived by matching the first two moments to the continuous distribution (see Appendix B)[3] to ensure convergence of the discrete and continuous models (see Rubinstein (2000) or Kamrad and Richtken (1991)):

$$p_{l,\kappa+1} = \frac{c^2\Delta t}{2(\Delta y)^2} + \frac{\eta^2}{2(\Delta y)^2} + \frac{\eta}{2\Delta y} \tag{23}$$

$$p_{l,\kappa} = 1 - \frac{c^2\Delta t}{(\Delta y)^2} - \frac{\eta^2}{(\Delta y)^2} \tag{24}$$

$$p_{l,\kappa-1} = \frac{c^2\Delta t}{2(\Delta y)^2} + \frac{\eta^2}{2(\Delta y)^2} - \frac{\eta}{2\Delta y} \tag{25}$$

where:

$$\eta = \mu(l,t)\Delta t + (l-\kappa)\Delta y$$
$$\mu(l,t) = a(b - y(l,t))$$

See Hull and White (1990) for a proof. When Δy is set to $c\sqrt{3\Delta t}$ the following dynamic rules for the choice of κ can be implemented to ensure positive probabilities (see Appendix C):

$$\kappa = \begin{cases} l+1 & \text{if } \dfrac{\mu(l,t)\Delta t}{\Delta y} \geq \sqrt{\dfrac{2}{3}} \\ l & \text{if } -\sqrt{\dfrac{2}{3}} < \dfrac{\mu(l,t)\Delta t}{\Delta y} < \sqrt{\dfrac{2}{3}} \\ l-1 & \text{if } \dfrac{\mu(l,t)\Delta t}{\Delta y} \leq -\sqrt{\dfrac{2}{3}} \end{cases} \tag{26}$$

These dynamic rules for the choice of κ imply minimum and maximum values for $y(l,t)$:

$$-\sqrt{\frac{2}{3}} \leq a(b - y(l,t))\frac{\Delta t}{\Delta y} \leq \sqrt{\frac{2}{3}}$$
$$\Leftrightarrow y_{\min} = b - \sqrt{\frac{2}{3}}\frac{\Delta y}{a\Delta t} \leq y(l,t) \leq b + \sqrt{\frac{2}{3}}\frac{\Delta y}{a\Delta t} = y_{\max}$$

The branching method is changed to $\kappa = l-1$ at a node (n,t), where n is the largest integer with $y = y(0) + n\Delta y \leq y_{\max}$ and to $\kappa = l+1$ at a node (m,t), where m is the smallest integer with $y = y(0) + m\Delta y \geq y_{\min}$.

[3]The probabilities could also be derived by converting the underlying differential equation into a set of difference equations by the explicit finite difference method. In this case the η^2 terms can be skipped. However, the procedure with the quadratic terms ensured better numerical convergence when we tested it.

As $y(l, t)$ has a range of $(-\infty, \infty)$ we impose the following restrictions on the product ab:

$$y_{\min} = b - \sqrt{\frac{2}{3}}\frac{\Delta y}{a\Delta t} \ll 0 \Leftrightarrow \sqrt{2}\frac{c}{\sqrt{\Delta t}} \gg ab \tag{27}$$

$$y_{\max} = b + \sqrt{\frac{2}{3}}\frac{\Delta y}{a\Delta t} \gg 0 \Leftrightarrow -\sqrt{2}\frac{c}{\sqrt{\Delta t}} \ll ab \tag{28}$$

5 BINOMIAL TREE MODEL FOR TWO ASSETS WITH STOCHASTIC CORRELATION

To approximate the system proposed in Section 2 we combine the two tree models introduced in Sections 3 and 4. The nodes in the combined tree are denoted by (i, j, l, t), where i and j indicate the number of up or down moves of the first and the second asset, respectively, and l specifies the level of the correlation that influences the probability structure of the movements of the assets in $t + 1$. As the correlations are not constrained in the binomial tree model in Section 3 the transformation (2) and the process for the transformation (3) of the stochastic correlation do not have to be restricted, and the tree approximations for the processes of two constantly correlated assets and for the stochastic correlation can be combined without any restriction. The two trees are arranged successively in such a way that the correlations $\rho_{l,t}$ resulting from the approximation of the stochastic correlation in time step t have an impact on the probabilities for an up or down jump of the assets in $t + 1$. The probabilities derived for the movements of the assets (22) also apply in the case of stochastic correlation. Furthermore, as we assume the Brownian motions of the underlying processes of the assets and of the transform of the correlation are independent, their probabilities can simply be multiplied to obtain the joint probability. Thus, a particular node branches into 12 different nodes in the next time step.

The nodes and their probabilities are specified in Table 1. The first column encloses all 12 possible branches from a single node (i, j, l, t), while the second column provides the probability of getting to the particular node as the product of $p_{x,y}$ and $p_{l,y,t}$ (where $x \in [a, b, c, d]$ and $y \in [l - 1, l, l + 1]$). The structure of the tree is illustrated in Figure 5, where the matrices in the second part of the figure describe the possible values of S_1 and S_2.

6 SENSITIVITY ANALYSIS AND COMPARISON WITH THE HULL TWO-DIMENSIONAL CONSTANT CORRELATION MODEL

Pricing a spread option in this framework involves a considerable number of input parameters. In the following, we want to stress the influence of the parameters of the stochastic correlation on the price of a spread option with a payoff $\max(S_1 - S_2 - K, 0)$. Each of the results is presented in two parts. First a particular case, called the "basic scenario" is defined and explored. In this scenario we compare our stochastic correlation model (SC model) with the Hull two-dimensional constant correlation

TABLE 1 Nodes and probabilities of the combined tree.

Nodes	Probability
$(i+1,\ j+1,\ l+1,\ t+1)$	$p_{a,l+1} \cdot p_{l,l+1,t}$
$(i+1,\ j-1,\ l+1,\ t+1)$	$p_{b,l+1} \cdot p_{l,l+1,t}$
$(i-1,\ j+1,\ l+1,\ t+1)$	$p_{c,l+1} \cdot p_{l,l+1,t}$
$(i-1,\ j-1,\ l+1,\ t+1)$	$p_{d,l+1} \cdot p_{l,l+1,t}$
$(i+1,\ j+1,\ l,\ t+1)$	$p_{a,l} \cdot p_{l,l,t}$
$(i+1,\ j-1,\ l,\ t+1)$	$p_{b,l} \cdot p_{l,l,t}$
$(i-1,\ j+1,\ l,\ t+1)$	$p_{c,l} \cdot p_{l,l,t}$
$(i-1,\ j-1,\ l,\ t+1)$	$p_{d,l} \cdot p_{l,l,t}$
$(i+1,\ j+1,\ l-1,\ t+1)$	$p_{a,l-1} \cdot p_{l,l-1,t}$
$(i+1,\ j-1,\ l-1,\ t+1)$	$p_{b,l-1} \cdot p_{l,l-1,t}$
$(i-1,\ j+1,\ l-1,\ t+1)$	$p_{c,l-1} \cdot p_{l,l-1,t}$
$(i-1,\ j-1,\ l-1,\ t+1)$	$p_{d,l-1} \cdot p_{l,l-1,t}$

model (CC model) and show that the CC model overprices the spread option and the correlation hedge parameter in the case of stochastic correlation.

Then a family of scenarios is studied, called the "global scenarios", which allows us to cover a large region of the parametric space $(\sigma_1, \sigma_2, a, b, c)$.

In the basic scenarios:

- $r = 0.04$, maturity $T = 1$ year;
- $\Delta t = 1/n$, $n = 70$ (number of time steps);
- $S_0^1 = 1$, $S_0^2 = 1$, $K = 0$;
- $\sigma_1 = 0.3$, $\sigma_2 = 0.13$;
- $y(t = 0) = b$, $b = \ln(\ln(2))$ (ie, $\rho = 0$), mean reversion speed $a = 1$, $c = 0.2$ (volatility of y).

In the global scenarios:

- r, T, Δ_t, S_0^1, S_0^2 as before;
- $K \in \{0, 1\}$, $\sigma_i \in 0.2, 0.4, 0.6, 0.8$, $i = 1, 2$;
- $a \in \{0.5, 1, 1.5\}$ with ($a_1 = 0.5$, $a_2 = 1$, $a_3 = 1.5$);
- $b \in \{\ln(\ln(4/3)), \ln(\ln(8/3)), \ln(\ln(4))\}$, with $b_1 = \ln(\ln(4/3))$ (ie, $\rho = -0.5$), $b_2 = \ln(\ln(8/3))$ (ie, $\rho = 0.25$) and $b_3 = \ln(\ln(4))$ (ie, $\rho = 0.5$);
- $c \in \{0.2, 0.4, 0.6, 0.8\}$ with $c_1 = 0.2$, $c_2 = 0.4$, $c_3 = 0.6$ and $c_4 = 0.8$.

6.1 Prices and numerical convergence

In this section we provide values of the option in different scenarios varying the number of time steps n, from $n = 10$ to 70. An estimate for the error is calculated by subtracting the values found for the different time steps from the value computed with $n = 70$.

For the basic scenario the value of the option is computed in the case of stochastic correlation and compared with the price assuming constant correlation. The corresponding estimated errors are illustrated in Figure 9. One can see from

FIGURE 5 Structure of the combined tree.

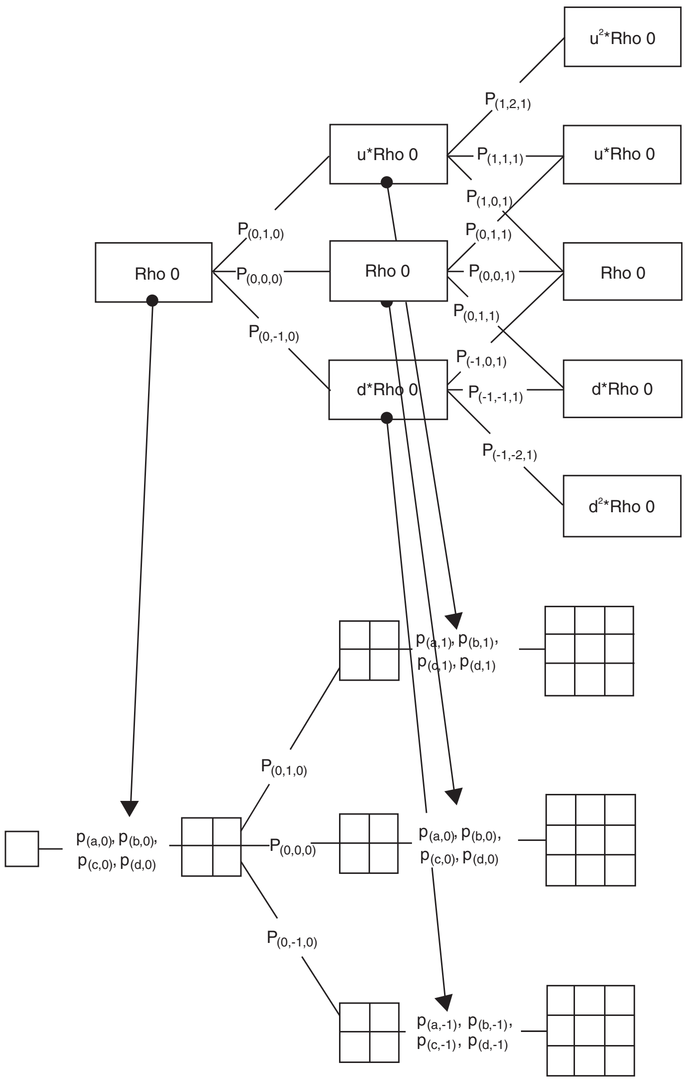


Figure 7 that from about 50 time steps up the price can be indicated with an accuracy of 10^{-4}.

It can easily be seen that in this case the CC model does not converge much quicker than the SC model. Subtracting the estimated errors from each other allows us to state that from 30 time steps the performance of both models is equal when considering computational convergence (see Figure 8).

Figure 9 shows the effect of increasing the number of time steps on the error in all global scenarios in a joint graph. In Figure 10 the same result is presented

FIGURE 6 Effect of varying the number of computational time steps on the estimated errors in the CC model.

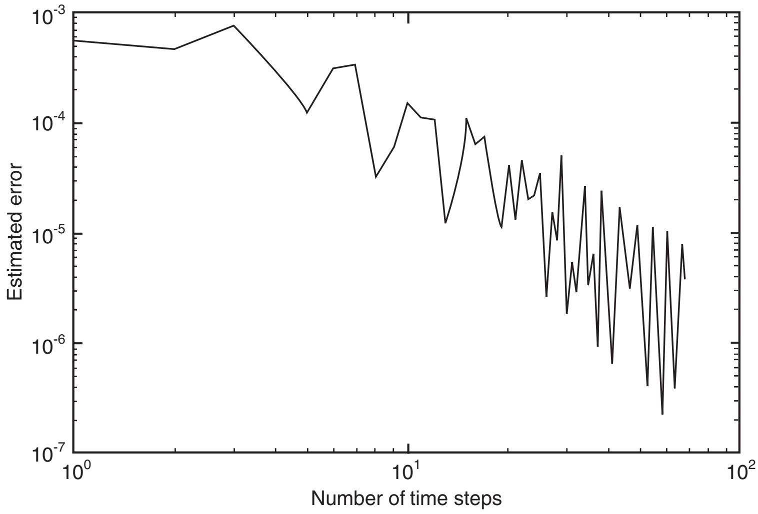


FIGURE 7 Effect of varying the number of computational time steps on the estimated errors in the SC model.

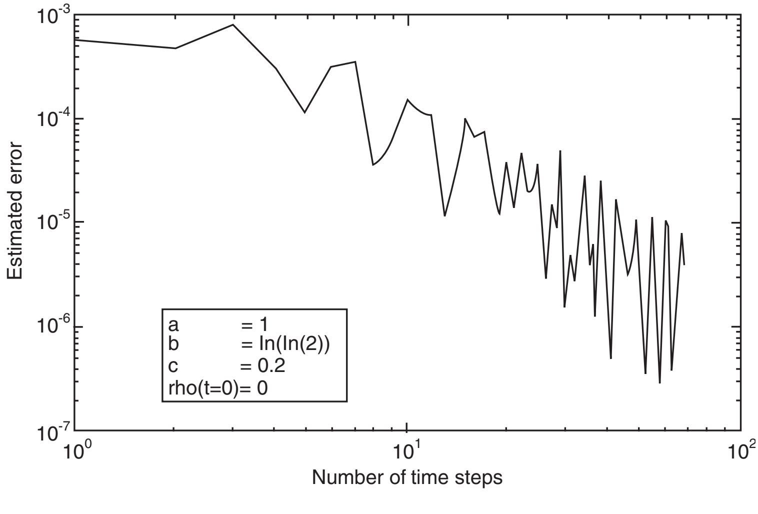


FIGURE 8 Difference between the estimated errors of the CC model and the SC model.

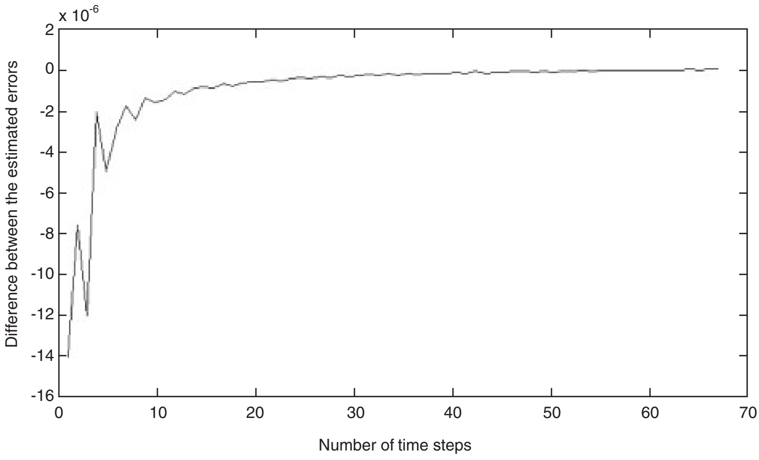
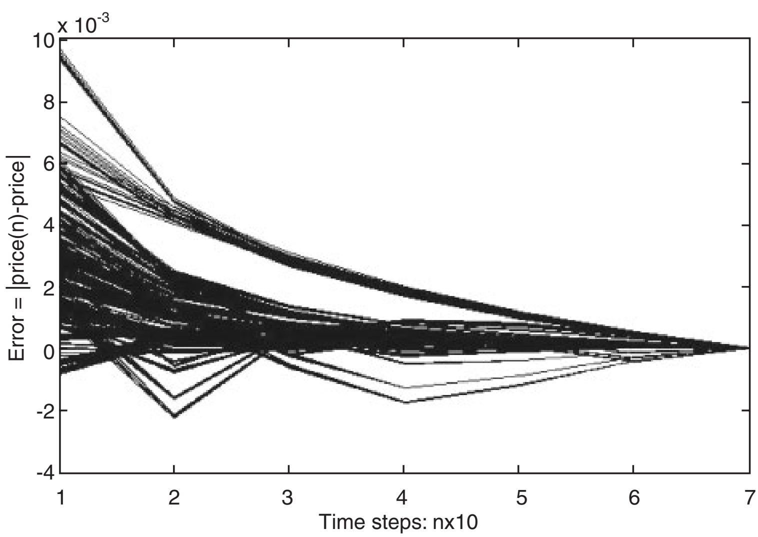


FIGURE 9 Global scenarios, convergence of prices.

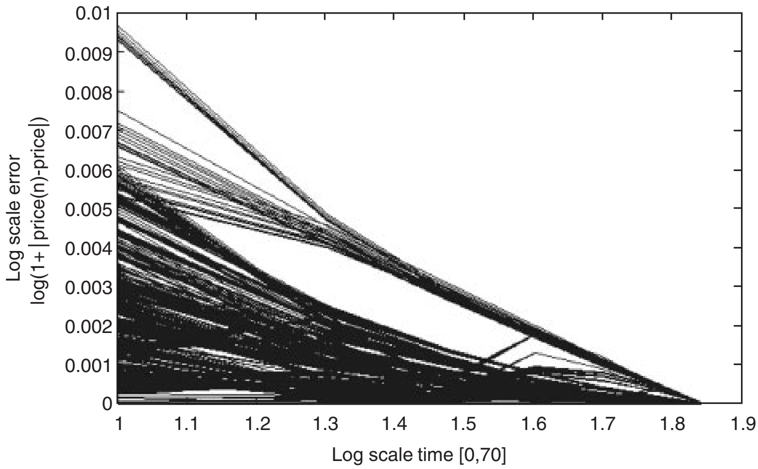


with a log–log scale. The actual set of prices (in bold characters) together with the convergence rates (abbreviated as conv and in light characters), computed by performing a log–log linear regression, is provided in Table 2. It should be noted that the intercept remained approximately constant throughout the global scenarios.

TABLE 2 Effect of varying parameters on the prices and convergence for the global scenarios. The first part uses $(\sigma_1, \sigma_2) = (0.8, 0.8)$, the second part uses $(\sigma_1, \sigma_2) = (0.8, 0.2)$ and the third part uses $(\sigma_1, \sigma_2) = (0.2, 0.2)$.

	b1 a1	**b1 a2**	**b1 a3**	**b2 a1**	**b2 a2**	**b2 a3**	**b3 a1**	**b3 a2**	**b3 a3**
				Large stock volatilities: (0.8, 0.8)					
c1	**0.07138**	**0.07140**	**0.07142**	**0.03530**	**0.03531**	**0.03532**	**0.02152**	**0.02150**	**0.02149**
conv	1.74858	1.80768	1.82662	1.41320	1.41316	1.41314	1.26662	1.26595	1.26547
c2	**0.07106**	**0.07116**	**0.07122**	**0.03519**	**0.03523**	**0.03525**	**0.02181**	**0.02172**	**0.02167**
conv	1.74956	1.81075	1.83055	1.41381	1.41341	1.41324	1.27627	1.27365	1.27169
c3	**0.07055**	**0.07076**	**0.07090**	**0.03509**	**0.03513**	**0.03517**	**0.02229**	**0.02210**	**0.02197**
conv	1.71837	1.80892	1.76955	1.41624	1.41475	1.41399	1.29161	1.28678	1.28270
c4	**0.06985**	**0.07021**	**0.07045**	**0.03503**	**0.03507**	**0.03511**	**0.02293**	**0.02262**	**0.02240**
conv	1.69125	1.79237	1.74070	1.42132	1.41803	1.41607	1.30949	1.30395	1.29837
				Large versus small stock volatilities: (0.8, 0.2)					
c1	**0.467034**	**0.467131**	**0.467197**	**0.319009**	**0.319147**	**0.319241**	**0.249142**	**0.249211**	**0.249258**
conv	1.60618	1.606622	1.606852	1.539939	1.540925	1.541441	1.479414	1.480281	1.480739
c2	**0.465773**	**0.466162**	**0.466424**	**0.317593**	**0.318093**	**0.318442**	**0.248858**	**0.249072**	**0.24923**
conv	1.604112	1.605894	1.606822	1.532328	1.536015	1.538019	1.469914	1.472982	1.47469
c3	**0.463691**	**0.464557**	**0.465142**	**0.31561**	**0.31656**	**0.317252**	**0.248739**	**0.249058**	**0.249321**
conv	1.600533	1.604568	1.606689	1.520505	1.527861	1.5321	1.456071	1.461692	1.46507
c4	**0.460843**	**0.462343**	**0.463368**	**0.313456**	**0.314812**	**0.315849**	**0.249033**	**0.249351**	**0.249658**
conv	1.595295	1.602471	1.606313	1.505879	1.516879	1.523681	1.440236	1.447869	1.452829
				Small stock volatilities: (0.2, 0.2)					
c1	**0.296391**	**0.296429**	**0.296454**	**0.241033**	**0.241049**	**0.241061**	**0.220044**	**0.220023**	**0.220008**
conv	1.711584	1.711967	1.712168	1.682873	1.683136	1.683277	1.662376	1.662226	1.662151
c2	**0.295904**	**0.296051**	**0.296151**	**0.240874**	**0.240927**	**0.240966**	**0.220384**	**0.220294**	**0.220234**
conv	1.709983	1.711509	1.712314	1.680948	1.681843	1.682357	1.661846	1.661165	1.660831
c3	**0.295111**	**0.295433**	**0.295652**	**0.240696**	**0.240776**	**0.240842**	**0.220957**	**0.220753**	**0.220616**
conv	1.707287	1.710674	1.712493	1.678242	1.679779	1.680762	1.661357	1.659669	1.658819
c4	**0.294047**	**0.294591**	**0.294968**	**0.240572**	**0.240652**	**0.240728**	**0.221735**	**0.221388**	**0.221148**
conv	1.7035	1.70936	1.712595	1.675268	1.677141	1.678512	1.66115	1.658048	1.656402

FIGURE 10 Global scenarios, convergence of prices on a log–log scale.



The values of the spread option with stochastic correlation converge at an average rate of $n^{-1.6}$. In Table 2 one can see that the convergence depends on the parametric region with an obviously lower rate of convergence for higher values of correlation ($y(0) = b$) and volatility of y (c). The convergence is worst for high stock volatilities and high correlation at the same time. In principle the results were not reliable for stock volatilities higher than 0.8 and stock correlations approximately outside of the region $[-0.95, 0.95]$. This high correlation could also be obtained using values of the mean reverting level b and/or the volatility c.

6.2 Correlation parameters

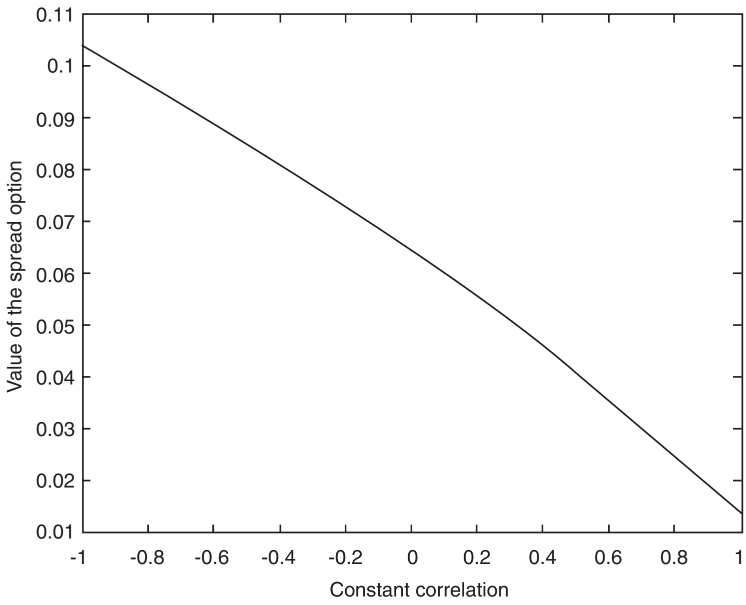
The correlation structure between the stocks has a substantial effect on the price of a spread option. This can already be seen with the CC model. In order to show the general relationship between correlation and the price of the option we analyze the effect of an increase in correlation on the price in the CC model. In Figure 11 the inverse relationship is illustrated: an increase in the correlation results in a lower spread option price. Furthermore, the slightly concave graph indicates that higher correlations have a larger impact on the price.

In the next step we want to break down the influence of the parameters of y on the spread option value in the SC model and compare this with the CC model. This is performed for the basic scenario.

Sensitivity of the price with respect to the volatility

In the SC model we set the mean reverting level b as well as the value of $y(0)$ in $t = 0$ to $\ln(\ln(2))$, which is equivalent to $\rho(0) = 0$, and vary the volatility of y, ie, c. The price of the spread option decreases with a rise in the volatility of the

FIGURE 11 Effect of varying correlation on the value of a spread option.



correlation (see Figure 12). Since higher correlations have a larger impact on the price, as we have seen before, an increase in the volatility of the mean reverting stochastic correlation causes a decrease in the prices of the derivative. In order to compare these results with the equivalent CC model we set the correlation $\rho \equiv 0$. The comparison of the two graphs shows that the CC model systematically overestimates the price of the option as the volatility of the correlation increases (see Figure 12).

Sensitivity of the price with respect to the mean reverting level

Analyzing the impact of the mean reverting level on the price of the spread option we find a similar effect. We vary b from $\ln(\ln(4/3))$, which corresponds to a mean reverting level for $\rho = -0.5$, to $\ln(\ln(4))$ (equivalent to a mean reverting level for $\rho = 0.5$) and set $\rho(0)$ to the respective mean reverting level value. All other parameters are kept constant. Figure 13 shows a negative interrelation between the values of the mean reverting level and the values of the spread options. The higher the long-term mean of the correlation the less probable it is that big spreads between the two shares will occur and therefore the value of the spread option has to fall with an increase in the mean reverting level.

In order to compare the results of the SC model to the CC model we compute the values of the CC model assuming that the constant correlation ρ is set to the mean reverting level for ρ_t in the SC model. Figure 13 visualizes that the CC model overstates the option values for negative long-term mean values the most, ie, higher

FIGURE 12 Effect of varying the volatility of y_t on the value of the spread option.

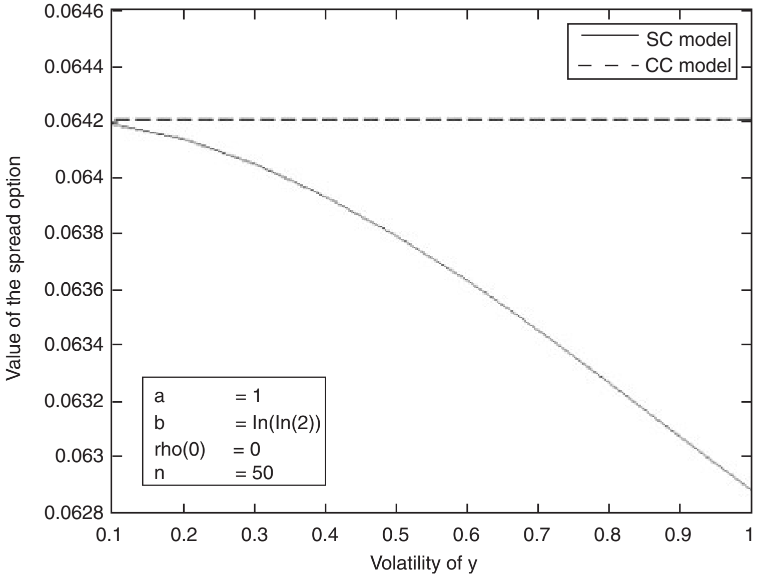


FIGURE 13 Effect of varying the value of the mean reverting level on the value of the spread option.

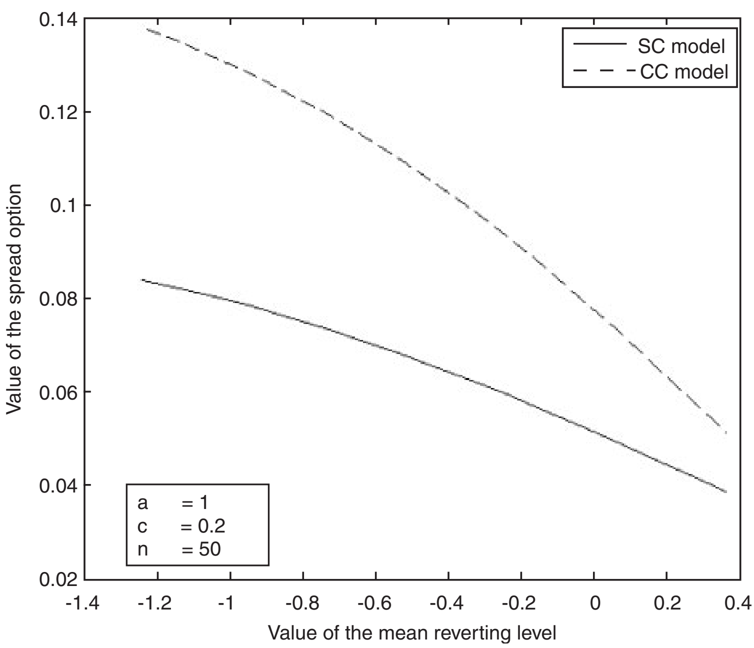


TABLE 3 Hedge ratios for chosen values.

Hedge ratio	In the money $S_1 = 1.6$	At the money $S_1 = 1$	Out of the money $S_1 = 0.4$
Δ_1	0.9449	0.5657	0.004
Δ_2	−0.8981	−0.4345	−0.0014
ν_1	0.1430	0.3677	0.0114
ν_2	0.0668	0.1996	0.0044
$\delta V/\delta\rho$	−0.013	−0.03	−0.0003

and lower correlations than the mean reverting level are possible. As the lowering effect of the highly positive correlations is greater (see Figure 11) the prices of the SC model are lower than those of the CC model. This effect is, of course, not as distinct for very positive mean reverting levels.

6.3 Hedge ratios

We compute the hedging parameters $\Delta_i = \partial V/\partial S_i(0)$, $\nu_i = \partial V/\partial\sigma_i$ and $\partial V/\partial\rho$, $i \in \{1, 2\}$, in $t = 0$, where V is the value of the spread option, which is dependent on S_i, σ_i and ρ_t. Here $V(S_i(0))$ $(V(\sigma), V(\rho))$ we denote the value of the option varying $S_i(0)$ (σ_i and ρ, respectively).

In Table 3 we provide these hedge ratios for sample values of S_1 in $t = 0$ for the basic scenario.

Delta hedge ratio

We compute the Δ sensitivity of the spread option by locally altering the value of S_1 and S_2 in $t = 0$, ie:

$$\text{Delta hedge ratio} = \frac{V(u_i \cdot S_i(0)) - V(d_i \cdot S_i(0))}{S_i(0) \cdot (u_i - d_i)}$$

where u_i, d_i are the up and down increments in the tree, respectively. This is a stable approach to computing derivatives on a tree (see Chung and Shackleton (2002)).

Figure 14 shows that the hedging parameter rises exponentially with an increase in S_1 for out-of-the-money options and with a decreasing radiant for in-the-money options and, thus, it corresponds to the Δ sensitivity of a plain vanilla call option. Figure 15 exhibits the exact opposite behavior: it falls exponentially for out-of-the-money options and with a decreasing radiant for in-the-money options. This appearance is due to the payoff structure of the spread option in T. The results for Δ_i are nearly the same in the constant correlation case as b is chosen equal to the start value of y.

Table 4 shows the Δ values for S_1 in the global scenarios. Note that the parameter a, the mean reversion speed, and c, the volatility of y, do not have a large impact on Δ. Here b has an influence on the value in a stronger way as we choose $y(0) = b$. The Δ values are smaller with higher correlation between the stocks. This is in accordance with constant correlation and may be explained by the fact that an increase in correlation decreases the variance of the spread $S_1 - S_2$.

FIGURE 14 Δ_1 as a function of the value of S_1 in t_0.

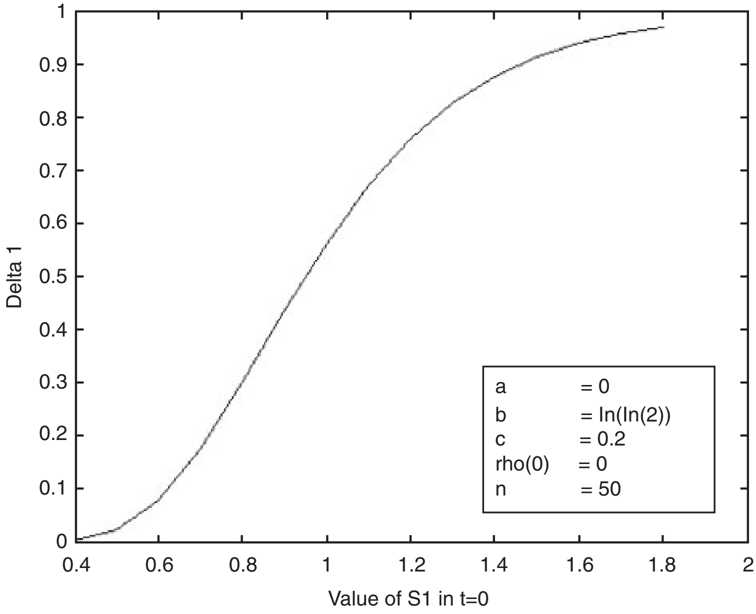


FIGURE 15 Δ_2 as a function of the value of S_1 in t_0.

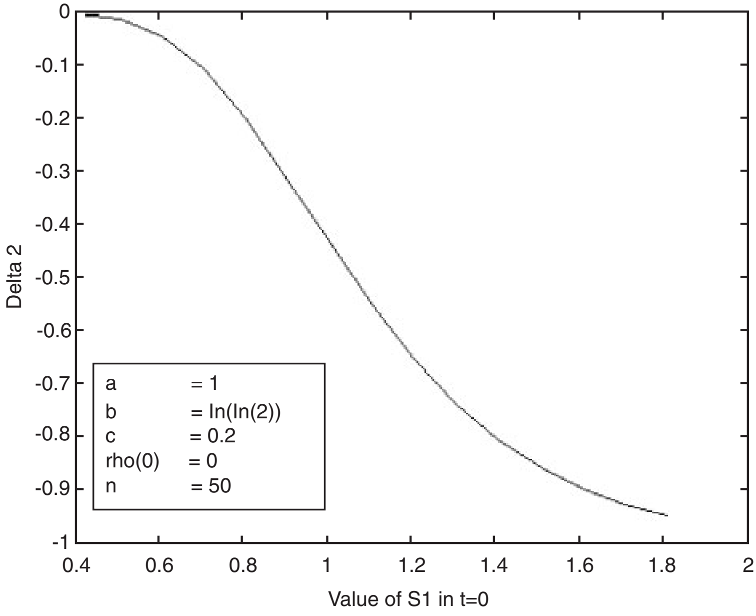
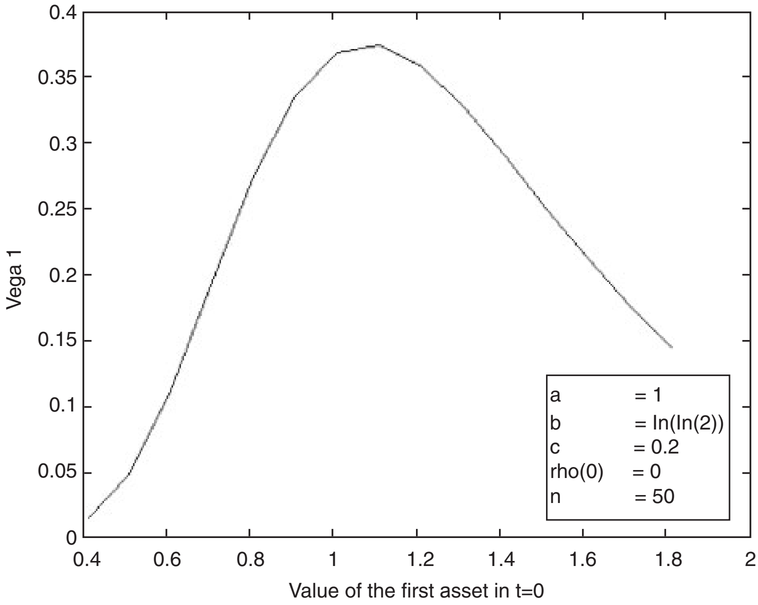


TABLE 4 The Δ values of S_1 in the global scenarios. The first part uses $(\sigma_1, \sigma_2) = (0.8, 0.8)$, the second part uses $(\sigma_1, \sigma_2) = (0.8, 0.2)$ and the third part uses $(\sigma_1, \sigma_2) = (0.2, 0.2)$.

	b1 a1	**b1 a2**	**b1 a3**	**b2 a1**	**b2 a2**	**b2 a3**	**b3 a1**	**b3 a2**	**b3 a3**
				Large stock volatilities: (0.8, 0.8)					
c1	0.362862	0.362922	0.362962	0.267523	0.267712	0.26784	0.209071	0.209279	0.209421
c2	0.3621	0.36234	0.362502	0.26555	0.266259	0.266747	0.20737	0.208073	0.208574
c3	0.360819	0.361366	0.361732	0.26263	0.264047	0.265055	0.205333	0.206534	0.207453
c4	0.359023	0.359998	0.360652	0.259252	0.261373	0.262952	0.203617	0.205129	0.206372
				Large versus small stock volatilities: (0.8, 0.2)					
c1	0.634557	0.634613	0.63465	0.546653	0.546796	0.546892	0.497116	0.497304	0.497431
c2	0.633835	0.63406	0.634211	0.545147	0.545692	0.546064	0.495444	0.496144	0.496627
c3	0.632634	0.63314	0.63348	0.542846	0.543975	0.544764	0.493037	0.494431	0.495456
c4	0.630969	0.631859	0.632461	0.540035	0.541819	0.543105	0.490339	0.492445	0.494009
				Small stock volatilities: (0.2, 0.2)					
c1	0.543041	0.543075	0.543098	0.491648	0.491689	0.491717	0.468353	0.468361	0.468367
c2	0.542601	0.542736	0.542827	0.491229	0.491376	0.491479	0.468411	0.468426	0.46844
c3	0.541876	0.542177	0.54238	0.490664	0.490933	0.491131	0.468603	0.468593	0.468599
c4	0.540886	0.541406	0.541762	0.490081	0.490447	0.490735	0.468974	0.468903	0.468875

FIGURE 16 ν_1 as a function of the value of S_1 in t_0.



Vega hedge ratios

We compute the Vega hedge ratio of the spread option by locally altering the value of σ_1 and σ_2 in $t = 0$, ie:

$$\text{Vega hedge ratio} = \frac{V(\sigma_i + tbp) - V(\sigma_i)}{tbp}$$

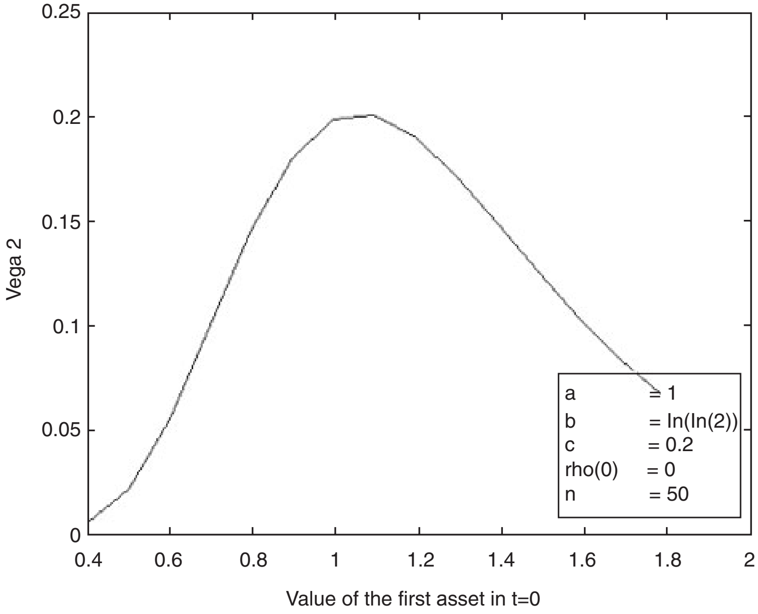
Note that the sensitivity of the spread option with respect to the volatility of the underlying assets increases for out-of-the-money options approaching the strike price, is highest for at-the-money options and decreases for in-the-money options (see Figures 16 and 17). As $\rho(0) = 0$ these results are in accordance with plain vanilla options, where the influence of the volatility is the greatest for options nearly at the money and at the money. As before the findings do not differ visibly from those we obtain for the Vega hedge ratio using the CC model because a and c are chosen relatively small in this example and we model the correlation independently of the variance structure of the underlying assets.

Table 5 shows the Vegas for S_1 in the global scenarios. An increase in the correlation $y(0)$ and b as well as the volatility of correlation c leads to a decrease in the Vegas. The first phenomenon also exists when assuming the constant correlation case owing to the decrease in the variance of the spread as a whole. An increase of a leads to the opposite reaction of Vega.

TABLE 5 The Vega values of S_1 in the global scenarios, considering three combinations of (σ_1, σ_2): (0.2, 0.2), (0.6, 0.2) and (0.6, 0.6).

	b1 a1	**b1 a2**	**b1 a3**	**b2 a1**	**b2 a2**	**b2 a3**	**b3 a1**	**b3 a2**	**b3 a3**
				Volatilities: (vol1, vol2) = (0.2, 0.2)					
c1	0.330952	0.357473	0.370336	0.353567	0.369488	0.378073	0.383729	0.385541	0.387474
c2	0.335323	0.360473	0.372831	0.358436	0.372328	0.381042	0.386033.	0.387146	0.386743
c3	0.342722	0.364488	0.378127	0.361556	0.37385	0.381433	0.386272	0.388299	0.385929
c4	0.346838	0.366876	0.3803	0.368402	0.376975	0.382292	0.386418	0.388445	0.385331
				Volatilities: (vol1, vol2) = (0.6, 0.2)					
c1	0.386849	0.384467	0.380731	0.383604	0.381112	0.375211	0.376531	0.372629	0.364474
c2	0.386628	0.384962	0.381837	0.382876	0.379992	0.376755	0.375504	0.370202	0.362947
c3	0.385734	0.383708	0.382018	0.382115	0.37836	0.375084	0.373078	0.367357	0.361013
c4	0.385113	0.382373	0.380351	0.381568	0.376416	0.372968	0.37219	0.364613	0.359227
				Volatilities: (vol1, vol2) = (0.6, 0.6)					
c1	0.212299	0.200434	0.19116	0.234513	0.229048	0.212883	0.226276	0.21692	0.208752
c2	0.208005	0.199718	0.187583	0.230048	0.226073	0.212038	0.225467	0.215984	0.20471
c3	0.206745	0.198384	0.188064	0.225675	0.221684	0.210375	0.222223	0.211009	0.203885
c4	0.206958	0.196816	0.189617	0.222671	0.217373	0.21005	0.217552	0.207845	0.199557

FIGURE 17 ν_2 as a function of the value of S_1 in t_0.



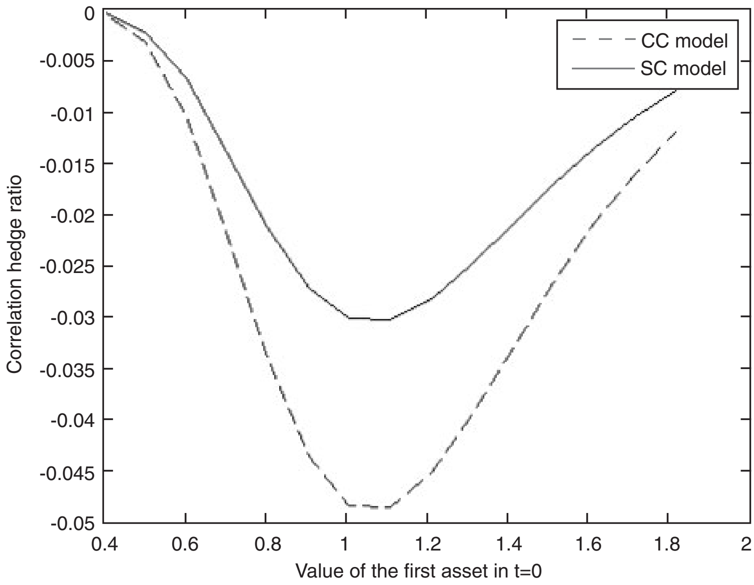
Correlation hedge ratios

We approximate the correlation hedge ratio $\delta V/\delta\rho$ by:

$$\frac{V(\rho(0)+df) - V(\rho(0)-df)}{2df}\frac{df}{d\rho}$$

where *df* represents the increments in the tree for the correlation process. Figure 18 reflects the negative relationship between correlation and the value of the spread option that we have already pointed out earlier. The hedge ratio falls for out-of-the-money options and increases for in-the-money options. The sensitivity with respect to the correlation is the highest for at-the-money options. To compute the respective hedge ratio for the CC model we varied the constant correlation. The hedge parameter computed in the CC model exhibits similar features (see Figure 18) as in the SC model. However, the CC model overestimates the sensitivity with respect to changes in the underlying correlation structure as it does not take into account the long-term mean. The correlation can be hedged with another instrument involving correlation or with another spread option on the same stocks but with different maturities. Table 6 shows the correlation hedging values for the global scenarios. Note that the hedge ratios are all negative. As we have stated before a higher correlation leads to a decrease in value of the spread option as an increase in correlation decreases the variance of the spread as a whole. The

FIGURE 18 The hedge ratios of the correlation as a function of the value of S_1 in t_0.



derivative of the correlation hedge ratio with respect to the correlation (b) seems to be slightly negative. The relationship between hedge ratio and c is negative, and the relationship between the hedge ratio and a is positive. So, as before, c seems to increase the terminal variance of the spread while a decreases it.

7 SUMMARY AND CONCLUSIONS

We have developed and implemented a tree model to price spread options on underlyings which are stochastically correlated based on a system of stochastic processes with a mean reverting process for the stochastic correlation. This model relaxes the constant correlation assumption in the existing literature. The tree model converges with an average error of $n^{-1.6}$ and the value of the spread option can be indicated with four digits computing more than 50 time steps. Thus, the convergence of the stochastic correlation model is as fast as the constant correlation tree model proposed by Hull (2002). Our framework allows us to examine several effects of a mean reverting stochastic correlation. We show that the equivalent constant correlation model overestimates the value of a spread option as well as the hedge ratio for a correlation hedge.

TABLE 6 The correlation hedge ratios of the stocks in the global scenarios. The first part uses $(\sigma_1, \sigma_2) = (0.8, 0.8)$, the second part uses $(\sigma_1, \sigma_2) = (0.8, 0.2)$ and the third part uses $(\sigma_1, \sigma_2) = (0.2, 0.2)$.

	b1 a1	**b1 a2**	**b1 a3**	**b2 a1**	**b2 a2**	**b2 a3**	**b3 a1**	**b3 a2**	**b3 a3**
				Large stock volatilities: (0.8, 0.8)					
c1	−0.03433	−0.02759	−0.02261	−0.04212	−0.03395	−0.02786	−0.04466	−0.03604	−0.02961
c2	−0.03461	−0.02776	−0.02271	−0.04125	−0.03343	−0.02753	−0.0431	−0.03511	−0.02901
c3	−0.03501	−0.02801	−0.02288	−0.03985	−0.03259	−0.02699	−0.0409	−0.03371	−0.0281
c4	−0.03544	−0.02831	−0.02308	−0.03805	−0.03146	−0.02626	−0.03845	−0.03206	−0.02697
				Large versus small stock volatilities: (0.8, 0.2)					
c1	−0.07297	−0.05864	−0.04804	−0.10201	−0.08215	−0.06738	−0.12145	−0.09791	−0.08036
c2	−0.07371	−0.05909	−0.04832	−0.1007	−0.08138	−0.06688	−0.11857	−0.09618	−0.07925
c3	−0.07483	−0.05977	−0.04876	−0.09845	−0.08005	−0.06603	−0.1141	−0.09343	−0.07746
c4	−0.07612	−0.06062	−0.04932	−0.09534	−0.07817	−0.06482	−0.10863	−0.08991	−0.07511
				Small stock volatilities: (0.2, 0.2)					
c1	−0.04767	−0.03831	−0.03139	−0.06066	−0.04887	−0.04009	−0.0677	−0.05459	−0.04482
c2	−0.04808	−0.03856	−0.03155	−0.05964	−0.04826	−0.03971	−0.06592	−0.05352	−0.04413
c3	−0.04867	−0.03893	−0.03179	−0.05798	−0.04727	−0.03907	−0.0633	−0.05188	−0.04305
c4	−0.04933	−0.03938	−0.03209	−0.05584	−0.04593	−0.0382	−0.06023	−0.04987	−0.04169

APPENDIX A PROOF OF PROPOSITION 1

Basic equations for the bidimensional binomial approximation are as follows:

$$p_1 S_1 u_1 + (1 - p_1) S_1 d_1 = S e^{r \Delta t} \tag{A.1}$$

$$p_2 S_2 u_2 + (1 - p_2) S_2 d_2 = S e^{r \Delta t} \tag{A.2}$$

$$(u_1 - 1)^2 p_1 + (1 - p_1)(d_1 - 1)^2 - (p_1(u_1 - 1) + (1 - p_1)(d_1 - 1))^2 = \sigma_1^2 \Delta t \tag{A.3}$$

$$(u_2 - 1)^2 p_2 + (1 - p_2)(d_2 - 1)^2 - (p_2(u_2 - 1) + (1 - p_2)(d_2 - 1))^2 = \sigma_2^2 \Delta t \tag{A.4}$$

$$\begin{aligned}(u_1 - 1)(u_2 - 1) p_a &+ (u_1 - 1)(d_2 - 1) p_b + (d_1 - 1)(u_2 - 1) p_c \\ &+ (d_1 - 1)(d_2 - 1) p_d - (u_1 - 1) p_1 + (d_1 - 1)(1 - p_1)(u_2 - 1) p_2 \\ &+ (d_2 - 1)(1 - p_2) = \sigma_1 \sigma_2 \rho \Delta t\end{aligned} \tag{A.5}$$

$$p_a + p_b + p_c + p_d = 1 \tag{A.6}$$

$$p_a + p_b = p_1 \tag{A.7}$$

$$p_d + p_c = 1 - p_1 \tag{A.8}$$

$$p_b + p_d = 1 - p_2 \tag{A.9}$$

$$p_a + p_c = p_2 \tag{A.10}$$

- Equations (A.1) and (A.2). The expectation of S_t in the tree has to meet the expectation of S_t in continuous time: $E[S_t] = S(0)e^{rt}$. The approximation is exact in this case.
- Equations (A.3) and (A.4). Here $Var[S_t] = S_0^2 e^{2rt}(e^{\sigma^2 t} - 1)$. However, for reasons of simplification we use the fact that $Var[\mathrm{d}S/S] = \sigma^2 \, \mathrm{d}t$, which implies:[4]

$$Var\left[\frac{\Delta S}{S}\right] = \sigma^2 \Delta t$$

- Equation (A.5). The same simplification is used for the covariance and the correlation.
- Equation (A.6). The probabilities of the four branches have to sum up to one.
- Equations (A.7), (A.8), (A.9), (A.10) are derived from the marginal probabilities of a single asset.

Reformulate (15) as follows:

$$\begin{aligned}u_1 u_2 (p_a - p_1 p_2) &+ u_1 d_2 (p_b - p_1(1 - p_2)) + d_1 u_2 (p_c - (1 - p_1) p_2) \\ &+ d_1 d_2 (p_d - (1 - p_1)(1 - p_2)) - \sigma_1 \sigma_2 \rho \Delta t = 0\end{aligned} \tag{A.11}$$

[4]This simplification has already been used by Hull (2002).

From (6)–(10) we obtain:

$$p_a = p_1 - p_b \tag{A.12}$$

$$p_b \quad \text{free} \tag{A.13}$$

$$p_c = (1 - p_1) - p_d = -p_1 + p_2 + p_b \tag{A.14}$$

$$p_d = (1 - p_2) - p_b \tag{A.15}$$

Substituting these expressions into (A.11) and solving for p_b leads to:

$$p_b = p_1(1 - p_2) + \frac{\sigma_1\sigma_2\rho\Delta t}{(d_2 - u_2)(u_1 - d_1)} \tag{A.16}$$

Substituting (A.16) into (A.12), (A.14) and (A.15) we obtain:

$$\begin{aligned} p_a &= p_1 p_2 - \frac{\sigma_1\sigma_2\rho\Delta t}{(d_2 - u_2)(u_1 - d_1)} \\ p_c &= p_2(1 - p_1) + \frac{\sigma_1\sigma_2\rho\Delta t}{(d_2 - u_2)(u_1 - d_1)} \\ p_d &= (1 - p_1)(1 - p_2) - \frac{\sigma_1\sigma_2\rho\Delta t}{(d_2 - u_2)(u_1 - d_1)} \end{aligned} \tag{A.17}$$

It follows from (11) for u_1, d_1 and for u_2, d_2 that:

$$u_1 - d_1 = \frac{e^{r\Delta t} - d_1}{p_1}, \quad u_2 - d_2 = \frac{e^{r\Delta t} - d_2}{p_2} \tag{A.18}$$

Substituting (A.18) into (A.17) we obtain:

$$p_a = p_1 p_2 - \frac{\sigma_1\sigma_2\Delta t p_1 p_2 \rho}{(e^{r\Delta t} - d_1)(d_2 - e^{r\Delta t})} \tag{A.19}$$

$$p_b = p_1(1 - p_2) + \frac{\sigma_1\sigma_2\Delta t p_1 p_2 \rho}{(e^{r\Delta t} - d_1)(d_2 - e^{r\Delta t})} \tag{A.20}$$

$$p_c = p_2(1 - p_1) + \frac{\sigma_1\sigma_2\Delta t p_1 p_2 \rho}{(e^{r\Delta t} - d_1)(d_2 - e^{r\Delta t})} \tag{A.21}$$

$$p_d = (1 - p_1)(1 - p_2) - \frac{\sigma_1\sigma_2\Delta t p_1 p_2 \rho}{(e^{r\Delta t} - d_1)(d_2 - e^{r\Delta t})} \tag{A.22}$$

Hence, ρ is restricted by the conditions for the probabilities ($0 \leq p_i \leq 1$):

$$p_a:$$

$$\frac{(p_1 p_2 - 1)(e^{r\Delta t} - d_1)(d_2 - e^{r\Delta t})}{\sigma_1\sigma_2 p_1 p_2 \Delta t} \leq \rho \leq \frac{(e^{r\Delta t} - d_1)(d_2 - e^{r\Delta t})}{\sigma_1\sigma_2\Delta t}$$

p_b :

$$\frac{p_1(p_2-1)(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{\sigma_1\sigma_2 p_1 p_2\Delta t} \leq \rho \leq \frac{(1-p_1(1-p_2))(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{p_1 p_2\sigma_1\sigma_2\Delta t}$$

p_c :

$$\frac{p_2(1-p_1)(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{\sigma_1\sigma_2 p_1 p_2\Delta t} \leq \rho \leq \frac{(1-p_2(1-p_1))(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{p_1 p_2\sigma_1\sigma_2\Delta t}$$

p_d :

$$\frac{((1-p_1)(1-p_2)-1)(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{\sigma_1\sigma_2 p_1 p_2\Delta t} \leq \rho \leq \frac{(1-p_1)(1-p_2)(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{p_1 p_2\sigma_1\sigma_2\Delta t}$$

APPENDIX B GENERAL TRINOMIAL TREE PROBABILITIES

In the trinomial tree the probabilities of $y(l, t)$ moving to $y(\kappa-1, t)$, $y(\kappa, t)$ and $y(\kappa+1, t)$ are chosen to match the first and second moments of the three-point jump process of the change in $y(l, t)$ to the continuous distribution. Thus, the following equations must be satisfied:

$$p_{l,\kappa-1}(\kappa-1-l)\Delta y + p_{l,\kappa}(\kappa-l)\Delta y + p_{l,\kappa+1}(\kappa+1-l)\Delta y = \mu\Delta t \tag{B.1}$$

$$p_{l,\kappa-1}(\kappa-1-l)^2\Delta y^2 + p_{l,\kappa}(\kappa-l)^2\Delta y^2 + p_{l,\kappa+1}(\kappa+1-l)^2\Delta y^2 - (\mu\Delta t)^2 = c^2\Delta t \tag{B.2}$$

$$p_{l,\kappa-1} + p_{l,\kappa} + p_{l,\kappa-1} = 1 \tag{B.3}$$

where:

$$\mu = a(b-y)$$

It follows from (B.3) that:

$$p_{l,\kappa} = 1 - p_{l,\kappa-1} - p_{l,\kappa+1} \tag{B.4}$$

Substituting (B.4) into (B.1) and reformulating it we obtain ($\mu(l, t) := \mu$):

$$-p_{l,\kappa-1}\Delta y + (\kappa-l)\Delta y + p_{l,\kappa+1}\Delta y = \mu\Delta t \tag{B.5}$$

which is equivalent to:

$$p_{l,\kappa+1} = \frac{\mu\Delta t}{\Delta y} - (\kappa-l) + p_{l,\kappa-1} \tag{B.6}$$

Substituting (B.4) and (B.6) into (B.2) we have:

$$\begin{aligned} &p_{l,\kappa-1}(\kappa-1-l)^2(\Delta y)^2 \\ &\quad+\left(1-p_{l,\kappa-1}-\frac{\mu\Delta t}{\Delta y}+(\kappa-l)-p_{l,\kappa-1}\right)(\kappa-l)^2(\Delta y)^2 \\ &\quad+\left(\frac{\mu\Delta t}{\Delta y}-(\kappa-l)+p_{l,\kappa-1}\right)(\kappa+1-l)^2(\Delta y)^2=\mu^2(\Delta t)^2+c^2\Delta t \end{aligned}$$

which is equivalent to:

$$\begin{aligned} p_{l,\kappa-1}=&\frac{c^2\Delta t}{2(\Delta y)^2}+\frac{\mu^2(\Delta t)^2+2(l-\kappa)\Delta y\mu\Delta t+(l-\kappa)^2(\Delta y)^2}{2(\Delta y)^2} \\ &-\frac{\mu\Delta t+(l-\kappa)\Delta y}{2\Delta y} \end{aligned} \tag{B.7}$$

With $\eta=\mu\Delta t+(l-\kappa)\Delta y$:

$$p_{l,\kappa-1}=\frac{c^2\Delta t}{2(\Delta y)^2}+\frac{\eta^2}{2(\Delta y)^2}-\frac{\eta}{2\Delta y}$$

Substituting (B.7) into (B.6) we obtain:

$$p_{l,\kappa+1}=\frac{c^2\Delta t}{2(\Delta y)^2}+\frac{\eta^2}{2(\Delta y)^2}+\frac{\eta}{2\Delta y} \tag{B.8}$$

Substituting (B.7) and (B.8) into (B.4) we obtain:

$$p_{l,\kappa}=1-\frac{c^2\Delta t}{(\Delta y)^2}-\frac{\eta^2}{(\Delta y)^2} \tag{B.9}$$

APPENDIX C SPECIFIC CHOICE OF TREE PROBABILITIES

As Hull and White do not provide the proof for this dynamic rule in their paper, the restrictions are shown for the example $\kappa=l$.

In the case of $\kappa=l$ the probabilities are:

$$p_{l,l+1}=\frac{c^2\Delta t}{2(\Delta y)^2}+\frac{\mu^2(\Delta t)^2}{2(\Delta y)^2}+\frac{\mu\Delta t}{2\Delta y} \tag{C.1}$$

$$p_{l,l}=1-\frac{c^2\Delta t}{(\Delta y)^2}-\frac{\mu^2(\Delta t)^2}{(\Delta y)^2} \tag{C.2}$$

$$p_{l,l-1}=\frac{c^2\Delta t}{2(\Delta y)^2}+\frac{\mu^2(\Delta t)^2}{2(\Delta y)^2}-\frac{\mu\Delta t}{2\Delta y} \tag{C.3}$$

To obtain positive probabilities, which are smaller than one, we have to ensure that (C.1)–(C.3) are positive:

$$p_{l,l+1}=\frac{c^2\Delta t}{2(\Delta y)^2}+\frac{\mu^2(\Delta t)^2}{2(\Delta y)^2}+\frac{\mu\Delta t}{2\Delta y}\geq 0\Leftrightarrow\frac{\mu^2(\Delta t)^2+\mu\Delta t\Delta y}{c^2\Delta t}\geq -1$$

Substituting $c^2\Delta t = \frac{1}{3}(\Delta y)^2$ results in:

$$\frac{\mu^2(\Delta t)^2 + \mu\Delta t\Delta y}{(\Delta y)^2} \geq -\frac{1}{3} \Leftrightarrow \left(\frac{\mu\Delta t}{\Delta y} + \frac{1}{2}\right)^2 + \frac{1}{12} \geq 0$$

which again does not impose any constraints on the parameters:

$$p_{l,l} = 1 - \frac{c^2\Delta t}{(\Delta y)^2} - \frac{\mu^2(\Delta t)^2}{(\Delta y)^2} \geq 0 \Leftrightarrow -\sqrt{\frac{2}{3}} \leq \frac{\mu\Delta t}{\Delta y} \leq \sqrt{\frac{2}{3}}$$

$$p_{l,l-1} = \frac{c^2\Delta t}{2(\Delta y)^2} + \frac{\mu^2(\Delta t)^2}{2(\Delta y)^2} - \frac{\mu\Delta t}{2\Delta y} \geq 0 \Leftrightarrow -\left(\frac{\mu\Delta t}{\Delta y} - \frac{1}{2}\right)^2 - \frac{1}{12} \leq 0$$

Thus,

$$-\sqrt{\frac{2}{3}} \leq \frac{\mu\Delta t}{\Delta y} \leq \sqrt{\frac{2}{3}}$$

Equivalently, for $k = l + 1$:

$$1 - \sqrt{\frac{2}{3}} \leq \frac{\mu\Delta t}{\Delta y} \leq \sqrt{\frac{2}{3}}$$

and for $k = l - 1$:

$$-\sqrt{\frac{2}{3}} \leq \frac{\mu\Delta t}{\Delta y} \leq -1 + \sqrt{\frac{2}{3}}$$

Considering all three branching methods the following dynamic rules for the choice of the parameter k can be derived:

$$k = \begin{cases} l+1 & \text{if } \dfrac{\mu\Delta t}{\Delta y} \geq \sqrt{\dfrac{2}{3}} \\ l & \text{if } -\sqrt{\dfrac{2}{3}} < \dfrac{\mu\Delta t}{\Delta y} < \sqrt{\dfrac{2}{3}} \\ l-1 & \text{if } \dfrac{\mu\Delta t}{\Delta y} \leq -\sqrt{\dfrac{2}{3}} \end{cases} \tag{C.4}$$

REFERENCES

Ball, C., and Torous, W. (2000). Stochastic correlation across international stock markets. *Journal of Empirical Finance* **7**(3–4), 373–388.

Brennan, M. J., and Schwartz, E. S. (1978). Finite difference method and jump processes arising in the pricing of contingent claims. *Journal of Financial Quantitative Analysis* **13**, 461–474.

Cox, J. C., *et al* (1979). Option pricing: a simplified approach. *Journal of Financial Economics* **7**, 229–263.

Carmona, R., and Durrleman, V. (2003). Pricing and hedging basket options in a log-normal model. Technical Report, Department of Operations Research and Financial Engineering, Princeton University.

Chung, S., and Shackleton, M. (2002). The binomial Black–Scholes model and the Greeks. *The Journal of Future Markets* **22**(2).

Collin-Dufresne, P., and Goldstein, R. (2001). Stochastic correlation and the relative pricing of caps and swaptions in a generalized-affine framework. Working Paper, Carnegie Mellon University.

Dempster, M. A. H., and Hong, S. S. G. (2000). Spread option valuation and the fast Fourier transform. Working Paper 26/2000, The Judge Institute of Management Studies, University of Cambridge.

Dupire, B. (1993). Arbitrage pricing with stochastic volatility. Working Paper, Banque Paribas.

Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies* **6**, 327–343.

Hull, J. (2002). *Options, Futures, and Other Derivatives*, 5th edn. Prenctice Hall, New York.

Hull, J., and White, A. (1990). Valuing derivative securities using the explicit finite difference method. *Journal of Financial and Quantitative Analysis* **25**(1), 87–100.

Hull, J. C., and White, A. (1996). *Hull–White on Derivatives*. Risk Books, London.

Jarrow, R. A., and Rudd, A. (1983). *Option Pricing (The Irwin Series in Finance)*. Irwin, Homewood, IL.

Kamrad, B., and Richtken, P. (1991). Multinomial approximating models for options with k state variables. *Management Science* **37**(12), 1640–1652.

Kaplanis, E. C. (1988). Stability and forecasting of the comovement measures of international stock market returns. *Journal of International Money and Finance* **7**(1), 63–75.

Makridakis, S. G., and Wheelwright, S. C. (1974). An analysis of the interrelationships among the major world stock exchanges. *The Economic Journal* **84**(333), 195–215.

Pearson, N. D. (1995). An efficient approach for pricing spread options. *Journal of Derivatives* **3**, 76–91.

Poitras, G. (1998). Spread options, exchange options, and arithmetic Brownian motion. *Journal of Futures Markets* **18**(5), 487–517.

Ramchmand, L., and Susmel, R. (1998). Volatility and cross correlation across major stock markets. *Journal of Empirical Finance* **5**, 397–416.

Rubinstein, M. (1994). Return to Oz. *Risk* **7**, 67–71.

Rubinstein, M. (2000). On the relation between binomial and trinomial option pricing models. *Journal of Derivatives* **8**(2), 47–50.

Shimko, D. C. (1994). Options on future spreads: hedging, speculation, and valuation. *Journal of Futures Markets* **14**(2), 183–213.

Shu, J., and Zhang, J. (2003). The relationship between implied volatility and realized volatility of S&P 500 Index. *Wilmott Magazine*, January, 83–91.

Stein, E. M., and Stein, J. C. (1991). Stock price distributions with stochastic volatility: an analytic approach. *The Review of Financial Studies* **4**(4), 727–752.

Zagst, R. (2002). *Interest Rate Management*. Springer Finance, Berlin.