The mathematics of risk transfer

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Abstract: In this paper we present a historical account of the evolution of mathematics and risk management over the last 20 years. In it, we will focus primarily on present credit market developments and we give an account of some new credit derivatives: collateralised fund obligations.

Keywords: mathematical finance; risk management; credit derivatives.

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1 Introduction: hedge funds of the 21st century

Canadian winters are extreme: cold and snow are a fact of everyday life. Canada spends over $1 Billion every year removing snow. As one example, consider the city of Montreal. The city spends over $50 Million every year removing snow, about 3% of its total budget. It does that through a fixed-price contract agreement with a third party, which starts on 15 November and ends in 15 April – the snow season. During this time, the city's exposure to snow removal costs are – to a large degree – predictable (see Flambard et al., 2007 for a detailed account). However, snow precipitation outside of this period can become very costly: it is outside of the contractual arrangement, and the city may incur into expenses which may, on a relative basis, exceed the ones during the snow season. The city is exposed to snow financial risk. But snow financial risk affects also other corporations, such as ski resorts. For them, the snow financial risk is opposite: low precipitation during the late part of the fall or early spring will yield operational losses compared to years when snow fall is ample early in the fall or late into the spring. They also face snow financial risk.

Sometime ago, a proposal was launched to partially mitigate this: a snow swap. In this, a city will pay a premium to a dealer when snow is scarce outside the snow season, and receive a premium if snow appears. Similarly, a ski resort will receive payments if snow is scarce and will pay if snow is plentiful. The dealer arranges this, and collects a commission for its services. The dealer has no risk exposure to snow precipitation because it is exchanging offsetting payments between the two parties. The snow swap did not succeed, however, because there was no agreement as to where the measurements for snow precipitation were to occur. The snow financial risk seemed to be solved by the snow swap, but the geographical spread risk could not be absorbed by anyone.

Let us consider the hypothetical following proposition: a group of investors (a fund) gets together, puts up some money upfront (merely as collateral), and decides to take the geographical spread risk. It will pay the city in the case of out-of-season snow falls in the city, and will pay the ski resort in case of no out-of-season snow falls at the resort. In contrast, it will receive payments from both if the opposite occurs. With a nominal payment of $1 million, and a nominal fee of 10% ($100,000), the deal will look as follows (Table 1).

Table 1 Cash flows for the snow swap

Payments	*Snow*	*No-snow*
City	–$1 million	$1 million
Ski resort	$1 million	–$1 million

The difference with the previous, unsuccessful snow swap is that in this case, both the city as well as the ski resort gets to measure the snow precipitation at the place of their choice, with the fund taking the geographical risk. To move ahead with our example, let us assume the snow events in both places are correlated at 50%, and the fund will charge a $200,000 fee for its risk: this means that the cash-flows for the fund will be (Table 2).

Table 2 Cash flows for the fund

Event	*Cash flow*	*Probability (%)*
Offset payments	$200,000	75
Pays both	–$1,800,000	12.5
Receives from both	$2,200,000	12.5

To get an idea of the quality of these funds, note that the expected return for the $2 million the fund had to invest to participate in the swap is $200,000 or 10%, comparable to an investment in the stock market. The standard deviation, however, is 50%, which is more or less comparable to a game of poker. From an investment viewpoint, this is not a very good proposition, as the risk is too high for the expected return. Things become more interesting if the fund decided to do similar swaps in other cities. If 100 independent swaps are considered, for a total of $200 million invested, the expected return continues to be 10%, but the standard deviation, as a measure of risk, now drops to 5%. As an investment, this is now better than investing in the stock market and the fund has a future.

But things are slightly better. In our snow fund, we raised $200 million to post as collateral for 100 different swap agreements. This was to give rise to an expected return of 10% ($20 million) for the period (six months), with a standard deviation of 5%. Note that in calculating our cash flows, we have neglected the fact that the collateral ($200 million) was not to be used except as a guarantee to the counterparties – cities and ski resorts – that our fund would be able to honour its payment obligations even when all deals may turn against the fund. In other words, the collateral is there just to enable the fund to have the right credit rating for the deal (see Moody 2003 for ratings). The fund would obtain a rating of AAA, the best possible. But there is no reason to hold the $200 million in cash, one could easily invest them in T-Bills (short-term interest notes issued by the government of the USA), and hence earn LIBOR, the ongoing risk-free interest rate. In this way, our return will be LIBOR + 10%, with a standard deviation almost unchanged.

Situations such as this one are becoming common at the beginning of the 21st century: a certain investment partnership takes on some risk, in an effort to obtain a return. The risk is often the result of providing risk mitigation to a third party, but the fund absorbed residual risk, which is often hard to deal with but it may be diversifiable, such as in our example. These funds, which often operate in areas where the traditional financial companies (banks, insurance companies, etc.) do not operate, and are sometimes based in domiciles which allow unregulated activities (Cayman, Bermuda, etc.), are generally called hedge funds.

But is this type of activity new? From an abstract point of view, financial activity is an affair in risk transfer. Stocks and bonds, the financial instruments of the 19th century, are designed to allow investors to participate in commercial enterprises; stock holders assume market risk, that is, the risk that the firm does not meet profitability expectations; bond investors are not exposed to that market risk and only assume default risk, that is, the risk that the issuing entity cannot meet its financial obligations. This is also called credit risk, and losses can also occur without the company defaulting: a mere credit downgrade will lead to a decrease in the market value of the bond, and hence a loss, realised or not.

In the latter part of the 20th century, market risk was traded massively through the derivatives market. Investors could buy price protection related to stocks, currencies, interest rates or commodities by purchasing options or other derivatives; some are standard, others are tailor-made and labelled 'over-the-counter'. At the same time, default (or credit) risk was considered through ad-hoc considerations, but was not part of a quantitative treatment, and hence risk transfer of credit risk was not common. Towards the end of the 20th century, events such as the Russian default, Enron and Worldcom and the demise of Long-Term Capital Management, put credit risk at the forefront of financial institutions, and credit transfer emerged.

Today, credit risk has been regulated in BIS-II, the resolution of the Bank for International Settlements, but the credit market has just started (although at the present time its volumes are very high). A host of new credit products are created everyday. Later in this paper, we will explore some of the newest ones, the Collateralised Fund Obligations, or CFOs, designed to provide financing to investors in hedge funds. What is interesting, from a mathematical viewpoint, is that the arrival of new credit-sensitive products that is accompanied with new risks, which need to be determined and priced.

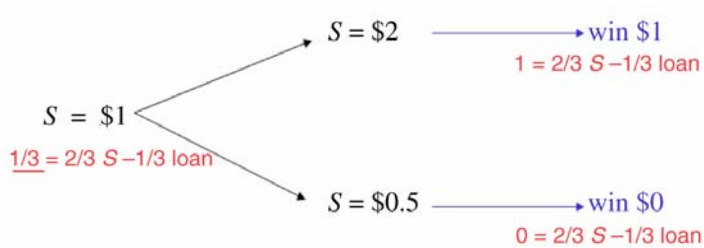
In this paper, we will review some of the earlier properties of financial risk, and we will focus on the analysis of CFOs as a means to highlight some of the new paradigms that we will likely to face in the near future.

2 Pricing risk

There are three types of risk: diversifiable risk, tradable risk (or hedgeable risk) and systemic risk. The first type of risk is the one we considered in the snow swap. There was nothing we could do to mitigate it, but building a portfolio of independent risks allowed us to diversify it to the point that it was worth taking. The second type of risk is tradable risk, best explained through the following example. The main difference with respect to our previous example is that, in this case, we will be able to price the risk accurately, as described below.

Imagine the following very simple hypothetical situation (see Figure 1).

Figure 1 Pricing fundamentals



There is an asset (a stock, a home, a currency, etc.) trading today at $1, which can only be worth $2 or $0.50 next year, with equal probability; interest rates are 0%, that is, borrowing is free. Consider also an investor, who may need to buy this asset next year and is therefore, concerned with increase in value; for that reason decides to buy insurance in the following form: if the asset raises to $2, then the insurance policy will pay $1. If the asset drops in price however, the policy pays nothing. This situation is summarised in Figure 1. One would be tempted to price this insurance policy with a premium obtained through probabilistic considerations, and it would seem that \$0.50 is the price that makes sense.

However, the following argument shows that this is not the case: if the investor paid \$0.50, then the seller of the policy could do implement the following investment strategy: he/she borrows an additional \$0.10, and buys 60% of the stock. If the stock raises in value, after paying the \$1 and returning the loan, he/she would make a profit of \$0.10. If, however, the stock drops in price, he/she will make a net profit of \$0.20, as the policy pays nothing and they only need to return the loan. In other words, \$0.50 is too much, as the issuer of the option will always make a profit: this phenomenon is called arbitrage, and it is a fundamental assumption for pricing theories that arbitrage should not exist (market design assumes that any chance of making free money will be eliminated from the market from smart traders, affecting the price which will immediately reach a non-arbitrage equilibrium). A simple calculation will show that the no-arbitrage price is exactly \$1/3. As opposed to traditional insurance premiums, financial insurance for tradable risks is not based merely on probabilistic considerations.

This simple example (a ‘call option’) is the basis of the no-arbitrage pricing theory (Hull and White, 2003), and we can quickly learn a few things from it. Firstly, the price of a contract that depends on market moves may be replicated with buy/sell strategies, which mimic the contract pay out, but can be carried out with fixed, predetermined costs. Secondly, there is a probability of events which is implied by their price, and is perhaps independent of historical events. In our example above, the implied probability of an up-move has to be 66%, and the probability of a down-move is 33%, because with those probabilities we can price the contract taking simple expectations. However, a more profound revision of the previous example will convince the reader with a background in diffusion processes that, if one takes the simple one-step example into a continuum of infinitesimal time/price increments, one ends with Brownian motion and associated Kolmogorov forward operator: the heat equation. One will also have a diffusion process for the asset or stock, and an associated diffusion process implied by market prices.

Black and Scholes (1973), derived the analogue of the heat equation and Brownian motion for the case of an option with an underlying stock price that follows the Ito process given by

$$\mathrm{d}S = \quad S\mathrm{d}t + \sigma S\mathrm{d}W^{\mathrm{P}}$$

where here S denotes the stock price,   denotes the drift, σ is the volatility, and dW are infinitesimal Brownian increments. An option on a stock is as a contract that will pay a future value at expiration: the payoff depends on the value of the underlying stock S, and will be denoted by $f_0(S)$. We denote by T the expiration time. Note the similarity with our simple example above (in Figure 1), the main difference being that in our case now the stock trades continuously and we could therefore, replicate our option by trading the stock continuously. In this case, the Black and Scholes theory shows that the price of the option contract is obtained by solving the following backward parabolic Partial Differential Equation (PDE), for all times $t < T$ prior to expiration.

$$\begin{cases} \dfrac{\partial f}{\partial t} + \dfrac{\sigma^2}{2} S^2 \dfrac{\partial^2 f}{\partial S^2} + rS\dfrac{\partial f}{\partial S} - rf = 0 \\ f(S,T) = f_0 \end{cases}$$

At first sight, this expression has two counterintuitive features: the absence of and the presence of the interest rate r in the PDE. A moment's reflection however, will convince us that this is not entirely surprising: after all, in our example in Figure 1 we already saw that the price of that option is independent of the probabilities of up and down moves of the stock, and it will only depend on the cost of borrowing. This was forced on us by our no-arbitrage assumption.

In more general terms, it turns out that option pricing can be established by taking expectations with respect to a 'risk neutral' measure Q, which is perhaps different from the historical measure P. In our particular case, this implies that the solution to the PDE is given by

$$f(S,t) = \frac{1}{\sqrt{2\pi(T-t)\sigma}} \int_0^{\alpha} f(u) \exp\left\{ -\frac{\left(\ln\left(u/S(t)\right) - \left(r - \left(\sigma^2/2\right)\right)(T-t)\right)^2}{2(T-t)\sigma^2} \right\} \mathrm{d}u$$

which is easily checked. From this perspective, pricing becomes equivalent to finding risk neutral probabilities and their pay-off expectations, and the PDE above is nothing but the Feynmann-Kac formula for this expectation.

The Black and Scholes theory also shows that one can replicate the option payoff by continuously trading the stock so that we always own - $\partial_S f$ units of it.

This signified a tremendous revolution, that won Black and Scholes the Nobel prize for Economics in 1997, as it not only established a pricing mechanism for the booming options and derivative markets, but because it established certainty where there was risk: derivatives could be replicated by buy/sell strategies with predetermined costs.

Their discovery revolutionised market risk perspectives. But Merton, who had rederived their pricing formalism using stochastic control theory, used this advance to start the modern theory of credit risk. His viewpoint, which we present below, was just as revolutionary.

Merton viewed a firm as shareholders and bond-holders. Bond-holders lent money to the firm, and the firm promised to pay back the loan, with interest. Shareholders own the value of the assets of the firm, minus the value of the debt (or liabilities); but firms have limited liability, which means that if the value of the assets falls below the value of the liabilities, in Merton's view, the firm defaults, shareholders owe nothing and the bondholders use the remaining value of the assets to recover a portion of their loan. In other words, the shareholders own a call option on the value of the assets of the firm, with a strike price given by the value of the liabilities at the given maturity time of the loan. The timing of his theory, which dates back to 1974, was perfect as the theory of option pricing had just been developed one year earlier, and this opened the ground for credit risk pricing and credit risk derivatives.

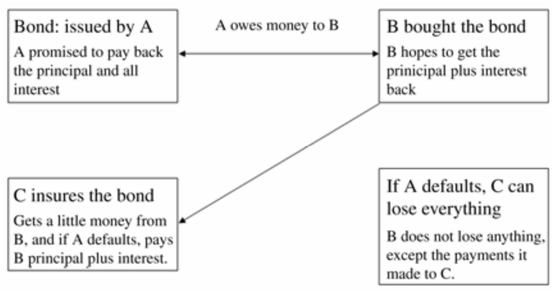
Strictly speaking, Merton approach assumes that the liabilities of a firm (its debt) expire at a certain time, and default could occur only at that time. Black and Cox (1976) conceptually refined Merton's proposal by allowing defaults to occur at anytime within the life of the option, creating the 'first passage default models'. The reason for this modification is that, according to Merton's model, the firm value could dwindle to nearly nothing without triggering a default until much later; all that matters was its level at debt maturity and this is clearly not in the interest of the bond holders. Bond indenture provisions therefore, often include safety covenants providing the bond investors with the right to reorganise or foreclose on the firm if the asset value hits some lower threshold for the first time. This threshold could be chosen as the firm's liabilities.

But the largest event in the credit market still had to wait until 1998, when the default of Russia and the menace of the impeachment of President Clinton over the Monica Lewinsky affair threw financial markets into disarray; the Russian default, and worries about the political stability of the USA created a credit crunch as bond investors fled from corporate debt for the more secure treasury bill market, introducing credit spread dislocations of historical proportions. This situation culminated with the collapse of Long-Term Capital Management, a multibillion dollar hedge fund that, anecdotally, had lured Scholes and Merton to their Board of Directors.

The result of these massive historical events was the explosion of the credit market. In it, financial players seek to buy and sell credit risk, either for insurance and protection in the case of default or bankruptcy of their counter-parties, or to take risk exposure which are considered either cheap or advantageous and therefore, earn, above average returns. The financial instruments, which are used in the credit market, are numerous, but two are especially noteworthy: Credit Default Swaps (CDS) and Collateralised Debt Obligations (CDO).

A CDS, (see Figure 2) is a contract that provides insurance against the risk of a default by particular company (known as the reference entity). The buyer of the insurance obtains the right to sell a particular bond issued by the company for its par value when a credit event occurs. The bond is known as the reference obligation and the total par value of the bond that can be sold is known as the swap's notional principal. The buyer of the CDS makes periodic payments to the seller until the end of the life of the CDS or until a credit event occurs. A credit event usually requires a final accrual payment by the buyer\newline.

Figure 2 A credit default swap



A CDO provides a way of creating securities with different risk characteristics from a portfolio of debt instruments. A general example would be, M types of securities are created from a portfolio of N bonds. The first tranche of securities has p_1 of the total bond principal and absorbs all credit losses from the portfolio during the life of the CDO until they have reached p_1 of the total bond principal. The second tranche has p_2 of the principal and absorbs all losses during the life of the CDO in excess of p_1 of the principal up to a maximum of $p_1 + p_2$ of the principal. The final tranche has p_M of the principal absorbs all losses in excess of $p_1 + p_2 + \ldots + p_{M-1}$ of the principal. The reason these instruments exist, is that, banks with large loan books, can use CDO's to effectively slice the default risk in those portfolios with credit-linked securities (the different tranches) and sell them to investors (who are often times hedge funds) in packets, which exhibit very different risk profiles: from the highly risk of the top – mezzanine – tranche (which will earn a higher fee spread), to the very secure last tranche, which will earn perhaps a minimal fee. One can also easily imagine similar situations where the underlying securities for the tranches are mortgages, not bonds. Many hedge funds are active participants as counterparties to these type of deals, and the hedge fund style that does this is called mortgage arbitrage (here, the term arbitrage is abused, in the sense that there is no real arbitrage, just a statistical arbitrage as the tranches pay more on average than other instruments with similar risk profiles).

The valuation of such structures is based on computing the probability distribution of the event '*m*th-default'. This is technically difficult because it requires one to handle the multivariate distribution of defaults and generally most credit models fail to reliably capture multiple defaults. There are basically two procedures for evaluating these basket derivatives, multifactor copula models similar to that used by researchers such as Li (2000) and Laurent and Gregory (2003) and Hull and White (2004) and intensity models (see Duffie and Garland, 2001).

In Escobar and Seco (2006), they present a PDE procedure for valuing a family of credit derivatives work within the structural framework, where the default event is associated to whether the minimum value of an stochastic processes (firm's asset value) have reached a benchmark, usually the firm's liabilities. More precisely, they assume:

the interest rate, r is constant

the value of the assets, $V_i(t)$, follows an Ito process with constant drift r and volatility $\sigma_i^2(t)$

$$\mathrm{d}V_i = rV_i\mathrm{d}t + \sigma_i(t)V_i\mathrm{d}W_i(t)$$

firm i defaults as soon as its asset value $V_i(t)$ reaches the liabilities, denoted as $D_i(t)$. This is the definition of default within the structural framework (see Black and Cox, 1976; Giesecke, 2003; Merton, 1974).

Define $X(t)$ = ln $V(t)$ as the n-dimensional Brownian motion vector with drift $= (\ _{1,\ldots,}\ _{n})$, $_i = r - \sigma_i^2(t)/2$ and covariances $\sigma_{i,j}(t)$. The running minimum is defined as:

$$\underline{X}_i(t) = \min_{0 \le s \le t} X_i(s)$$

They show that the price is a function of the multivariate density p of the vector of joint Brownian motions and Brownian minimums (it can be easily extended to maximums).

$$P\left(X_1(t) \in \mathrm{d}x_1, \ldots, X_n(t) \in \mathrm{d}x_n, \underline{X}_1(t) > m_1, \ldots, \underline{X}_n(t) > m_n\right)$$
$$= p\left(x_1, \ldots, x_n, t, m_1, \ldots, m_n, \quad , \Sigma\right) \mathrm{d}x_1, \ldots, \mathrm{d}x_n$$

For the case of more than two underlying components, p is the solution of a PDE with absorbing and boundary conditions (a Fokker-Planck equation) given by

$$\begin{cases} \dfrac{\partial p}{\partial t} = -\displaystyle\sum_{i=1}^{n} \ _i(t)\frac{\partial p}{\partial x_i} + \sum_{i,j=1}^{n} \frac{\sigma_{ij}(t)}{2}\frac{\partial^2 p}{\partial x_i \partial x_j} \\ p(x, t=0) = \Pi_{i=1}^{n}\delta\left(x_i\right) \\ p\left(x_1, \ldots, x_i = m_i, \ldots, x_n, t\right) = 0, \quad i = 1, \ldots, n \\ x_i > m_{i,} m_i \le 0, \quad i = 1, \ldots, n \end{cases}$$

In a one-dimensional setting, assuming constant drift and volatility, the solution is closely related to the inverse Gaussian distribution

$$p\left(X_1(t) \geq m_1\right) = \Phi\left(\frac{m_1 - \quad t}{\sigma\sqrt{t}}\right) - \exp\left\{\frac{2m_1}{\sigma^2}\right\}\Phi\left(\frac{-m_1 - \quad t}{\sigma\sqrt{t}}\right)$$

He et al. (1998) provided an explicit formula for the joint density for the case of two Brownian motions, or two underlying stocks. Formulas for the general *n*-dimensional case remain unknown.

3 Collateralised fund obligations

Let us consider now our next, and final, example, which will bring together the hedge fund example of Section 1, and the credit derivatives of the previous one.

There are over 10,000 hedge funds in the world. Many of them try to obtain returns independent of market directions but the majority of them, unlike our snow fund example, try to do it through financial instruments which are traded in the financial exchanges: equities, bonds, derivatives, futures, etc. They often try to extract return from situations of inefficiency: for example, they would buy a stock – also termed 'taking a long position' – which they perceive as undervalued with respect to the value of its underlying assets, and would sell short – borrow or 'take a short position' – a stock they perceive as overvalued with respect to the value of their assets, expecting a convergence to their fair price, hence obtaining return in the long term and not being subject to the direction of the stock markets, which will probably affect their long and short stock portfolio the same way. Others may do the same with bonds: there will be bonds which will earn slightly higher interest than others, simply because there are less numbers of them, and hence trade slightly cheaper than other bond issues, large and popular. Other funds, will monitor mergers between companies and try to benefit from the convergence in equity value and bond value that takes place after a merger by taking long and short positions in the companies' stocks and/or bonds. And we already mentioned, those funds, which try to benefit from the slightly higher interest earning properties of tranches of mortgage pools with respect to borrowing interest rates.

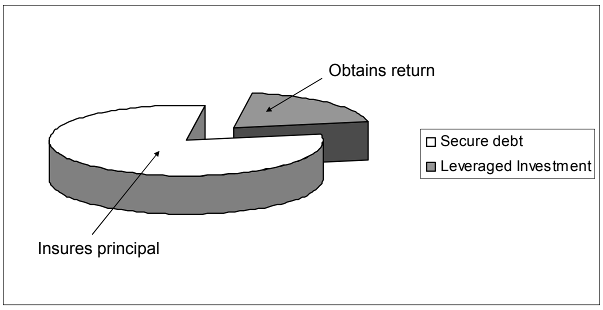
All of this gives investors with a wide universe to make investment choices. Let us imagine that each of those funds gives us returns similar to the snow fund: LIBOR + 10% expected return and 5% standard deviation. A portfolio of such investments will give us the same expected return, but the standard deviation is likely to decrease, because their return streams will be uncorrelated with each other. These investments, at least on paper, look extremely attractive. However, for the risk diversification to truly exist, one needs to invest in a sufficiently large number of them; there is always the possibility of fraud – these funds are largely unregulated and unsupervised, convergence-based trades may take a long time before they work, and deviations from our mathematical expectations may occur in the short term, etc. And, unlike stocks, or mutual funds, these funds often require minimum investments of the order of $1M. This means, that diversifying amongst them, will require substantial amounts of money.

There are several ways to invest in hedge funds: the three more frequent ones are:

Fund of funds: they are simple portfolio of hedge funds. The assets of the fund-of-funds are invested in a number of hedge funds (from 10 to 100). The chosen hedge funds are usually of a variety of different trading styles to achieve maximum diversification.

Leveraged products: imagine an investor has \$10 million to invest in hedge funds. Instead of allocating \$1 million to a portfolio of 10 different hedge funds, they may borrow an additional \$30 million from lenders, and invest the total amount \$40 million, in 40 different hedge funds. The investor pays interest to the lenders, and keeps the remaining gains. We will describe these type of investments in higher detail below.

Guaranteed products: they are term products, issued at maturities of five years, for example. The investor is guaranteed their money back after that period – five years – with no interest of course. In lieu of interest, they will receive a variable amount, which will be linked to the performance of the hedge fund portfolio. If the performance is good, the payment may be very large. If not, they simply get their money back, without interest. They are issued by a high-quality institution, who will take the investor's assets, invest a portion on a bond that will guarantee the principal at maturity of the note, and invest the rest – the interest earnings that the investor gives up – in a leveraged product we described earlier, to maximise the return of the investor's assets. They are very popular with retail products, as well as for institutions who can only invest in bonds, as these can be structured as a bond.



Anatomy of a guarantee

Leveraged products are attractive because of the following. Back in our snow swap example, expected return was LIBOR + 10%. LIBOR is the base lending rate. With proper collateral, lending at LIBOR + 1% is very feasible. That means that we can borrow at LIBOR + 1%, and invest at LIBOR + 10%. In other words, for every dollar we borrow we will make 9 cents for free, after paying all fees. Therefore, investors should want to borrow as much as possible and invest all the borrowed amounts. If it was not for the standard deviation, indeed that would be fantastic. The standard deviation, as well as other risks, limit the borrowing capacity and appetite of investors.

Leverage products are most often offered by banks; they lend to investors, investors take the first risk that the funds do not perform as expected, but the banks face the secondary risk that the losses exceed the equity provided by the investors and a portion of the lent amount may also be lost. Let us just mention that a number of safety measures are put in place by the banks to prevent this from happening, such as partial liquidation of the investments as the performance deviates from expectations. Recently, leverage products are organised by banks, but the borrowed amount is done through outside investors, through bond tranches very similar to the CDO structures we reviewed in our previous section. To explain how it works, we consider the case of the Diversified Strategies CFO SA, launched in 2002. Investors provided equity worth $66.3 million, which supported an investment of $250 million in hedge funds. The additional funds ($183.70 million) were raised through three bond tranche issues, as follows:

AAA tranche ($125 million)

a tranche ($32.5 million)

BBB tranche ($26.2 million).

We are not going to go in great detail into the details of the transaction; we will simply mention that the tranche structure is similar to a CDO; the bond investors provide the capital, and upon maturity, get their principal and interest. In case, the CFO structure fails to have enough assets to pay back its debts, the CFO will enter into default. In that scenario, the AAA-tranche investors are first in line to get their money back (principal plus interest). Next in line will be the A tranche, and the BBB tranche will be the last in line. In a default situation, the equity investors would have lost all their assets. Because of the difference in default risk, each of the bond investors receive different interest payments, highest for the BBB tranche, lowest for the AAA investors.

The interest payment for what their risk is worth –a credit spread- is a very interesting risk pricing problem. It is easier than the CDO pricing problem we described earlier, since here, we only need to look at the performance of the entire fund performance, and we do not need to enter into individual default numbers. In fact, with the assumption that the fund returns are normally distributed, it is very easy to determine the credit spread. The probability of default will be given by the quantile of a normally distributed Ito process, which has a simple risk-neutral analogue, and we just price that using expectation under the risk neutral measure. In the case of the Diversified Strategies CFO, the respective interest rates were as follows:

AAA tranche: LIBOR + 0.60%.

A tranche: LIBOR + 1.60%.

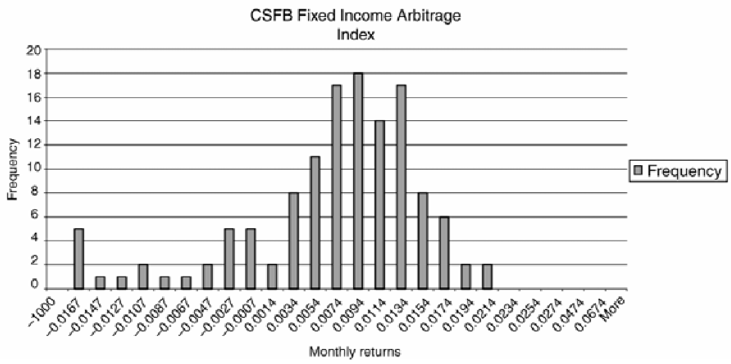
BBB tranche: LIBOR + 2.80%.

4 Non-GAUSSIAN returns

Many of the mathematical theories that study financial problems make a fundamental assumption: returns are normally – or lognormally – distributed. It is a reasonable assumption that permits robust mathematical modelling. However, non-Gaussian properties of real market data are a fact, and considerable effort goes into the mathematical modelling of such situations that relaxes the Gaussian assumptions.

In our context, the non-Gaussian nature of real markets exhibits itself in two main ways:

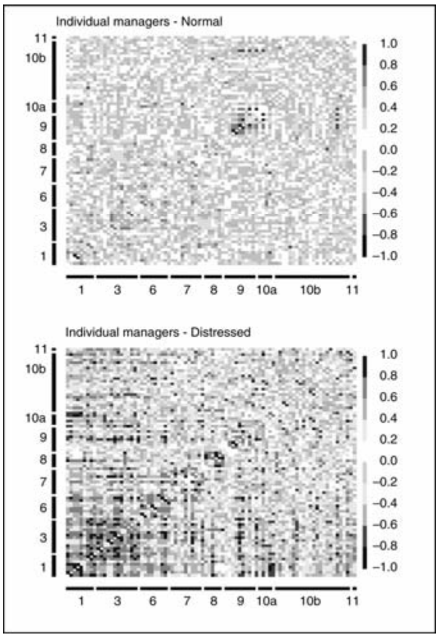
Non-Gaussian marginal distributions. The graph below depicts the monthly return frequency of a hedge fund index, the CSFB fixed income arbitrage index.



There are clear non-Gaussian features, for example, fat tails, also called Kurtosis, which in this case we can trace back to the events of 1998, and asymmetry, also known as skewness. This second feature comes naturally for most series, as return cannot go below –1, event of total lost, but still could theoretically be as positive as wanted. This left-bounded range, together with the drive of companies to emphasise above average growth, led to asymmetric distributions for the returns. Other common but difficult to graph marginal features are: time dependent return volatilities, trends in the return's mean as well as cycles, just to mention a few.

> Non-Gaussian dependence structures. If one tries to determine the dependence amongst several assets fitting it to a correlation matrix, one often finds that at certain times, the simultaneous occurrence of certain events does not correspond to a correlation measure.

This is a high-dimensional phenomenon, which is not so easy to describe graphically, but we will try to explain with the following sets of pictures.



In the first one, we see the correlation matrix of a hedge fund universe. The matrix is read from left to right, and from the bottom to the top, and number close to +1 or –1 are represented with a dark pixel, whereas numbers close to 0 are represented with a light pixel. We see that correlations are mostly low, with few instances of high correlations. This is consistent with our view of hedge funds.

The second picture represents the correlations taking into account only months of unusual returns; say months where the returns exceed the Gaussian safety band of two standard deviations from the mean. We see a very different correlation structure, with increased high correlation numbers.

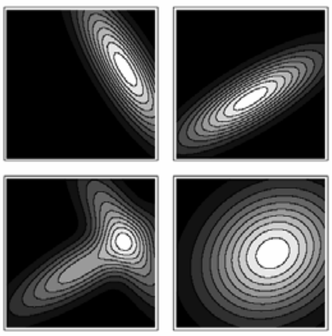
We denote this as correlation risk, or correlation breakdown phenomena. Given that correlation is one of the fundamental properties of hedge fund investing – remember our snow fund, correlation breakdown is a very damaging non-Gaussian effect for hedge fund portfolios and related structures.

This previous presentation assumed correlation as the right measure to describe dependence. The very emphasise in the correlation as ‘the measure’ to describe dependence structures has been strongly challenged since 1990s by the mathematically more general notion of Copulas (see Harry, 1997), for which the Gaussian correlation is a particular case. This area of research is quite complex from a mathematical viewpoint and at the same time is difficult to provide a meaning and a reliable estimation framework to the various parameters that appear; therefore, it is still very much under development.

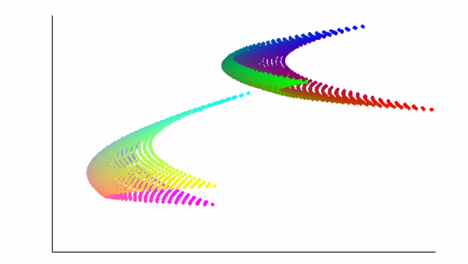
These non-Gaussian dependence structure features have an important impact all over mathematical finance, leading to interesting results on apparently unrelated issues like Portfolio theory and Derivative Pricing.

On the former, Buckley et al. (in press) studied the implications for Portfolio Theory of assuming multidimensional Gaussian-mixture distributions for the underlying returns.

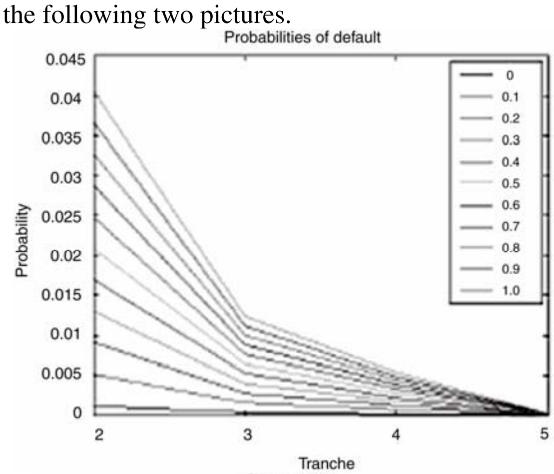
The following feature shows contour plots of probability density functions when working with multidimensional Gaussian Mixtures. The top row contains two bivariate Gaussian distributions potentially for the tranquil (left) and distressed (right) regimes. The bottom row illustrates the composite Gaussian mixture distribution obtained by mixing the two distributions from the top row (left) and a bivariate normal distribution with the same means and variance/covariance matrix as the composite (right).

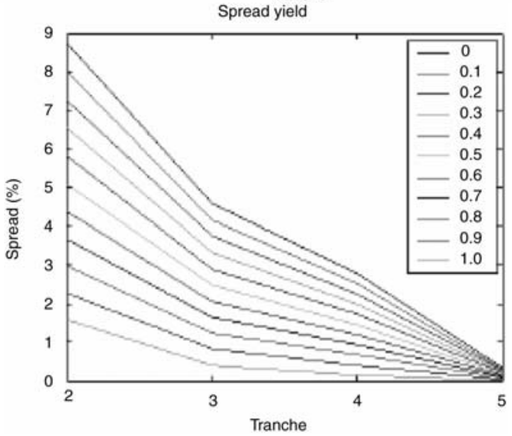


Investment opportunity sets for the tranquil and distressed regimes superimposed onto the same plot. The axes are the portfolio mean and variance. Typically the Gaussian Mixture approach optimal portfolio will be suboptimal with respect to both the tranquil and distressed mean-variance objectives. This is a three asset example.



On the latter, its effect on the default probabilities, and associated credit spreads for CFO tranches has been studied in Ansejo et al. (2006). More precisely, it is shown there that the credit ratings of CFO tranches are sensitive to the correlation breakdown probability, as summarised by the following two pictures.





The first figure shows that the probabilities of default spread over a substantial range when changing the probability of a distress month $1–p$ (market conditions). For example, the mezzanine tranche probability of default could go from 2% to 9%. In picture 2, we report the sensitivities of the spread yield to the market condition parameter p, which present a similar behaviour to probabilities of default.

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