Correlation Breakdown in the Valuation of Collateralized Fund Obligations

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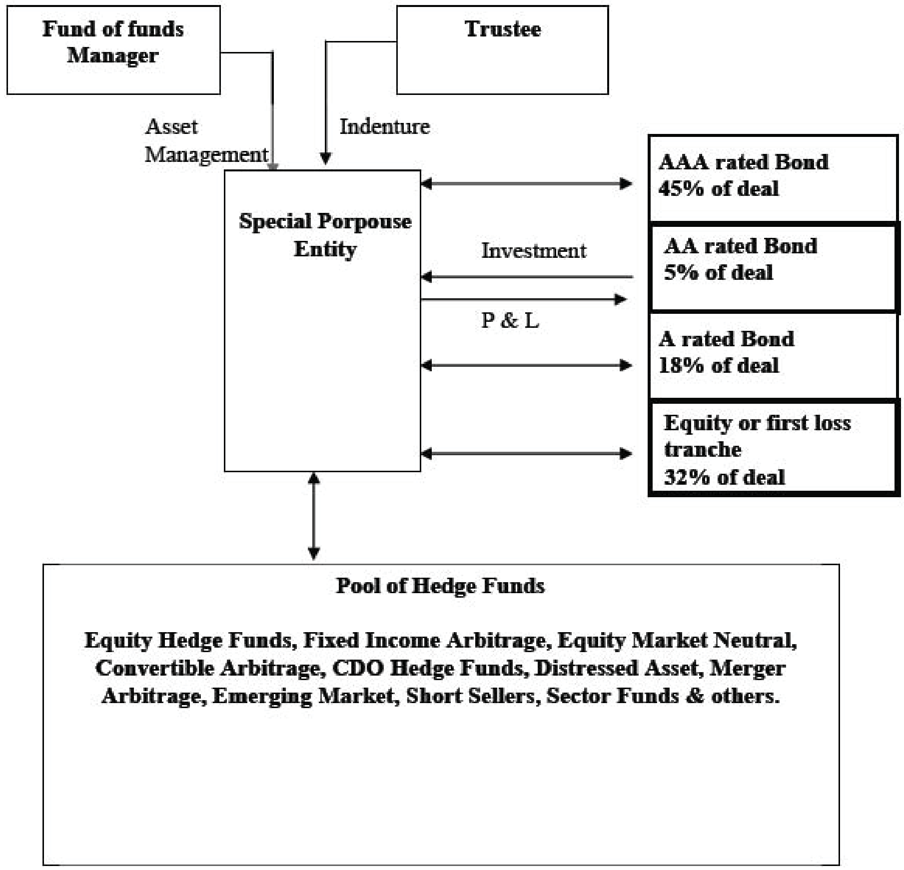
Collateralized Debt Obligations (CDO) are structured credit vehicles that redistribute credit risk to meet investor demands for a wide range of rated securities with scheduled interest and principal payments. CDOs are securitized by diversified pools of debt instruments. Recent developments in credit structuring technology include the introduction of Collateralized Fund Obligations (CFO).The capital structure of a Collateralized Fund Obligation is similar to traditional CDOs, meaning that investors are offered a spectrum of rated debt securities and equity interest. Although any managed fund can be the source of collateral, the target collateral in these structures tends to be hedge funds, such as relative value hedge funds, event-driven hedge funds or commodity trading advisors (CTAs), along with funds that finance the needs of growing companies, such as private equity and mezzanine funds. Often, a special purpose entity purchases the pool of underlying hedge fund investments, which are then used as collateral to back the notes.

These asset-backed notes are also called tranches. The most senior tranche is usually rated AAA and is credit-enhanced due to the subordination of lower tranches. This means that the lowest tranche, which is typically the equity tranche, absorbs losses first. When the equity tranche is exhausted, the next lowest tranche begins absorbing losses. A CFO may have an AAA-rated tranche, an AA-rated tranche, a single A tranche, a BBB rated tranche and an equity tranche. Exhibit 1 shows a schematic prototype of a CFO structure. A CFO can be regarded as a financial structure with equity investors and lenders where all the assets, equity and bond are invested in a portfolio of hedge funds. The lenders earn a spread over interest rates and the equity holders, usually the manager of the CFO, earn the total return of the fund minus the financing fees.

As pointed out in recent articles on this topic (Mahadevan and Schwartz [2002] and Stone and Zissu [2004]), both investors and issuers find CFO securitizations attractive. Investors, because a triple-A-rated bond will have a yield similar to that of a triple-A collateralized debt obligation plus a premium, because with this structure they gain exposure to a diverse collection of hedge funds through a fund of funds manager. And issuers, because securitization is a convenient vehicle for raising funds for an otherwise relatively illiquid product. Stone and Zissu [2004] provide a detailed overview of the first securitization of a fund of funds, the Diversified Strategies CFO SA, launched in June 2002. This CFO issued US$251 million in five tranches, rated AAA, AA, A, BBB and the equity. Also in June of 2002, lawyers from the Structured Products Group completed the second Collateralized Fund Obligation backed by hedge fund portfolios and assigned ratings

EXHIBIT 1

Schematic Prototype of a CFO Structure. Typical Maturities 3–7 Years



by Moody's Investors Service, Inc. and Standard & Poor's Rating Services. The CFO, titled Man Glenwood Alternative Strategies I or "MAST I," issued rated notes totaling US$374 million and nonrated notes and preference shares totaling US$176 million. As the first of their kind, these CFOs were viewed as a cutting edge transaction within the industry and attracted considerable attention in the financial press.

As far as valuation is concerned, Moody's in August 2003, began to use HedgeFund.net data to evaluate the risks of the underlying collateral in order to develop an accurate Montecarlo-based rating model for CFOs (see Moody's [2003]). Although CDOs have attracted much attention in the academic literature (Hull and White [2004]; Li [2000], and Laurent and Gregory [2003]), no effort has been made to develop an analytical rating model for CFOs.

The credit rating of a CFO tranche depends directly on the mark-to-market value of the pool of hedge funds. Collateralized fund obligations tend to be structured as arbitrage market value CDOs, meaning that the fund of funds manager focuses efforts on actively managing the fund to maximize total return while restraining price volatility within the guidelines of the structure. The diversity of hedge fund investment strategies and the active management of portfolio positions among strategies ensure that hedge fund risk and return characteristics are different from those of traditional assets (equities, bonds or ETFs), as illustrated by numerous articles in the literature (e.g., Moody's [2003] and CISDM [2006]). The unique return and risk characteristics of hedge funds are better characterized by more general distributions than the usual multivariate Gaussian approach. Therefore the probability of default of the different tranches will be also more volatile. In addition, lack of transparency within hedge funds in general makes it more difficult to obtain high-frequency historical data which leaves investors with no choice but to calibrate rather complex valuation models including the typical non-Gaussian behavior of hedge funds with monthly data.

With the ultimate aim of assessing the model and calibration risk incurred by CFOs, we develop in this article an analytical procedure for valuing the different tranches of a CFO by calculating the probability of default and the credit spreads from the bonds, and analyzing the pricing sensitivities to changes in parameters. When calculating the probabilities of default and the credit spreads of the different tranches of a CFO, one should take into consideration the non-Gaussian behavior of the returns from the pool of hedge funds. As we shall show later, those credit spreads are directly related to the percentiles of the underlying portfolio distribution.

In our pricing model we will assume that returns from the pool of hedge funds follow a Covariance-Switching (CS) Stochastic Process which is defined in the Appendix. This framework allows us to incorporate two well known characteristics from the return series of hedge funds: first, the skewed and leptokurtic nature of the marginal distribution functions, and second, the asymmetric correlation or correlation breakdown phenomenon (Longin and Solnik [2001]). As we will later prove, the correlation between the different hedge funds depends on the direction of the market. For instance, correlations tend to be larger in a bear market than in a bull market.

The structure of this article is as follows. Providing empirical evidence of the existence of correlation regimes, in Section 2 we present the Covariance-Switching process as the model for the profit and loss function for a pool of hedge funds and we outline some of their properties. The model for valuing a CFO is presented in Section 3, and in Section 4 we illustrate the strong parameter dependence with an example of a hypothetical CFO based on the S&P CTA index. This methodology can be applied to other daily hedge fund indexes such as the Dow Jones Hedge Fund Strategy Benchmarks and others. Section 5 contains our conclusions.

POOL OF HEDGE FUNDS MODEL

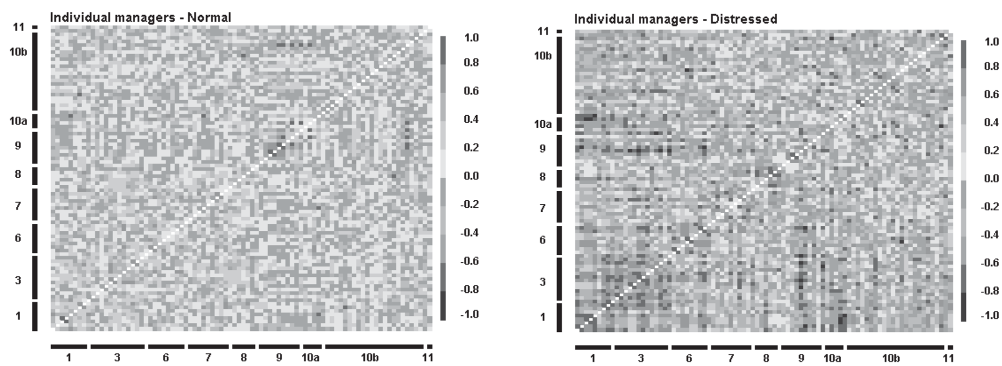
Hedge funds in general follow dynamic investment strategies using a variety of assets. Typically the investment process is not transparent and in most cases knowledge about future cash flows to be generated by the fund may be closely guarded. Hedge funds are largely unregulated because they are typically limited partnerships with fewer than 100 investors, which exempts them from the Investment Company Act of 1940. Offshore hedge funds are non-U.S. corporations and are not subject to SEC regulation. This limited regulation allows hedge funds to be extremely flexible in their investment options. Hedge funds can use short selling, leverage, derivatives, and highly concentrated investment positions to enhance returns or reduce systematic risk. They can also attempt to time the market by moving quickly across diverse asset categories. Hedge funds attract mainly institutions and wealthy individual investors, with minimum investments typically ranging from $250,000 to $1 million. Additionally, hedge funds often limit an investor's liquidity with lock-up periods of one year for initial investors and subsequent restrictions on withdrawals to certain intervals. Specific investments made by hedge funds are often carefully guarded secrets. This lack of transparency may make it difficult for a fund of hedge funds manager to assess the fund's aggregate exposure to a particular investment on a portfolio basis. It also presents a challenge to the fund manager attempting to monitor a particular hedge fund's adherence to its advertised style or investment approach. However in certain cases the transparency issues indicated above can be alleviated if the manager allows separate accounts. In order to account for the plethora of hedge fund and CTA strategies and the speed at which a given fund's composition can be changed, we model the alternative assets to exhibit more exotic distributions than the Gaussian one, displaying asymmetric and leptokurtic returns.

As is the case with most diversified portfolios, fund of hedge funds reduce the numerous types of risks arising from individual hedge funds by diversifying both across strategies as well as the number of funds in each strategy. Both types of diversification reduce risk, but because of the potential for style drift and a rapid total loss of value in a single fund, diversification across funds is especially important. Diversification is most effective for assets that exhibit low correlations in value over time. Fund of hedge funds' managers pursuing a low-volatility strategy seek to assemble a portfolio of relatively uncorrelated assets. Unfortunately, in times of market distress, such as the second half of 1998, previously uncorrelated hedge funds or CTAs can become highly correlated for short periods. This phenomenon is often called correlation breakdown. Evidence of this is presented in Exhibit 2, which shows

EXHIBIT 2

Correlation Switching Phenomenon

Each pixel represents a correlation number between the funds, one from each axis. As we can see, it is apparent that in distressed periods the correlations between the different managers tend to be greater in absolute value than in normal or tranquile periods.



the correlation matrices between the return series of hedge fund managers under tranquil and distressed regimes. During tranquil periods correlations are lower, whereas during periods of market distress, the asset returns become highly correlated, with the magnitudes of off-diagonal correlation values being close to one in absolute terms. Therefore, diversifying amongst different assets or markets in times of market distress was less effective at reducing risk than many participants had hitherto believed. Articles exploring the contagion phenomenon include Harvey and Viskanta [1994]; Longin and Solnik [2001], and Koedijk and Campbell [2002].

In order to capture both the unidimensional leptokurtic and asymmetric nature of hedge fund returns and the correlation breakdown phenomenon, we select the Covariance-Switching Stochastic Process (CS process) for modeling the returns of a pool of hedge funds from the range of parametric alternatives to a Geometric Brownian Process. With this election we develop a very tractable model (calculations using this often closely resemble those using a Brownian Process) and allow for a good adjustment to market data. Regime switching models have been used before in the field of finance but mainly in its univariate version (see Yao, Zhang and Zhou [2003] for option pricing and Choi [2004] for interest rate modeling). Moreover, regime switching models have the theoretical appeal that by adding together a sufficient number of components, any multivariate distribution may be approximated with reasonable accuracy. With an infinite number of contributions, any distribution can be reconstructed exactly.

A general CS is defined through its stochastic differential equation as follows:

Definition: $X_i(t)$ follow a covariance switching process with parameters $(p, \lambda, \mu, \mu_i^0, \mu_i^1, \sigma_i^{\cdot,0}, \sigma_i^{\cdot,1})$ if the diffusion process can be represented as:

$$dX_i(t) = \mu_i(t)dt + \sigma_i(t) \cdot dW(t) \tag{1}$$

$$\mu_i(t) = \mu + J_t \cdot \mu_i^0 + (1 - J_t) \cdot \mu_i^1$$

$$\sigma_i(t) = J_t \cdot \sigma_i^{\cdot,0} + (1 - J_t) \cdot \sigma_i^{\cdot,1}$$

$$\sigma_i^{\cdot,k}(t) = (\sigma_i^{1,k}(t), \ldots, \sigma_i^{n,k}(t))$$

Where J_t is a jump process (see appendix for details), $i = 1, \ldots n$, $j = 1, \ldots d$, W_t is a n-dimensional vector of independent Brownian motion processes, which are independent of J_t. Moreover, $\sigma_i^{j,k}$, μ_i^k are constants ($k = 0, 1$; $i = 1, .., n$; $j = 1, \ldots, n$).

In this work the vector of hedgefunds will be assumed $CS(p, \lambda, \mu, \sigma^{\cdot,0}, \sigma^{\cdot,1})$ under the historical measure (P–measure). Notice no jump in the drift is assumed, and then they are $CS(p, \lambda, r, \sigma^{\cdot,0}, \sigma^{\cdot,1})$ under the risk neutral measure (Q–measure). Therefore the returns will follow $CS(p, \lambda, \mu, \mu^{\cdot,0}, \mu^{\cdot,1}, \sigma^{\cdot,0}, \sigma^{\cdot,1})$ under P and $CS(p, \lambda, r, \mu^{\cdot,0}, \mu^{\cdot,1}, \sigma^{\cdot,0}, \sigma^{\cdot,1})$ under Q (see appendix for details).

Notice that the conditional distribution implied from a Covariance-Switching process is not closed form but a discrete approximation that leads to a Gaussian mixture distribution, providing an intuitive interpretation of CS. For example, taking an increment $\Delta t = 1$ then a discrete approximation to the CS density would be the density function of a mixture of two multivariate Gaussians:

$$f(\mathrm{x}) = \frac{p(1-e^{-\lambda})}{\sqrt{2\pi \det \Sigma_0}} e^{-\frac{1}{2}(\mathrm{x}-\mu_0)\Sigma_0^{-1}(\mathrm{x}-\mu_0)^t} + \frac{1-p(1-e^{-\lambda})}{\sqrt{2\pi \det \Sigma_1}} e^{-\frac{1}{2}(\mathrm{x}-\mu_1)\Sigma_1^{-1}(\mathrm{x}-\mu_1)^t} \qquad (2)$$

where $\mu_i = \mu + \mu^i$ and $\Sigma_i = \sigma^{\cdot,i} \cdot \sigma^{\cdot,i}$ are the i-th means vector and the i-th variance-covariance matrix. If the given Gaussian mixture GM distribution were used to describe the monthly returns of a portfolio, with the first Gaussian one could model the tranquil regimes and with the other the distressed ones. Moreover, the parameter p could be interpreted as the probability of having a tranquil month while $1 - p$ would be the probability of a distressed month. Unfortunately, GM is not the conditional distribution of a CS process (see appendix), the main difference between GM and the conditional distributions of CS being that for CS, p would be a function of time whose path affects the gaussianity of the components in the mixture.

Fitting parameters to a Covariance-Switching Process has not been explored in great detail in the literature. Few articles address the unidimensional case (see Choi [2004] and Chourdakis [2002]) but none, as far as we know, deals with multidimensional situations. Therefore we use an ad-hoc algorithm based on a multivariate test of gaussianity. The algorithm focuses on detecting the moments where a jump has occurred, and then uses the samples from tranquil and distress scenarios to estimate standard multidimensional Gaussian processes. As a first pass, we define a distressed month as a month where any of the returns of the portfolios' hedge funds is greater in absolute value than two standard deviations, and a tranquil month is when all of them are smaller. Then we add or remove sample points in order to create two groups where the hypothesis of multidimensional gaussianity is accepted.

CFO MODEL

In our proposed model we make the following assumptions. First, we consider a single period model, typically a year, in which distributions of random variables are sufficient to specify the model. Furthermore, we suppose that returns of the pool of hedge funds follow a Covariance-Switching process $CS(p, \lambda, r, \mu^{\cdot,0}, \mu^{\cdot,1}, \sigma^{\cdot,0}, \sigma^{\cdot,1})$. Second, we assume that there is no risk of default in the pool of hedge funds, therefore the portfolio's profit and loss function will be only market driven. This assumption is reasonable insofar as the fund of hedge funds does not contain a high proportion of speculative grade hedge funds, so the probability of default of each hedge fund is not relevant. Besides, introducing more complexity in the model will result in an increased number of parameters, and even with the simple model we propose, we already have large confidence intervals, as we show later.

One could adopt the more realistic but rather abstract view that many hedge funds are nothing but a junior tranche of a massive credit derivative, in recognition of their propensity to default (as was the case of Long Term Capital Management, Beacon Hill, and more recently, Amaranth, among others). From this viewpoint, the tranche of a CFO on a fund of hedge funds will be come a *second order* credit derivative, analog of a CDO squared. The analysis of this, to which this discussion will provide a frame of reference, is left to a later study. However, this *second derivative* nature of a fund of funds CFO only hightens the need to study non-gaussian dependence behavior among the hedge funds, as dependence will not behave linearly in the tails of the distribution.

Let us introduce some notation:

- S_i^0 and S_i are the initial and end values of the i-th hedge fund in the pool, $1 < i < n$.
- π and π_0 are the initial and end values of the portfolio.
- R_i is the return of the i-th hedge fund.
- n_i is the number of units invested in the i-th hedge fund and θ_i is the monetary quantity invested, so that $\theta_i = n_i S_i^0$.
- $PL = \pi - \pi_0$ is the profit and loss of the portfolio and $R_{PL} = \frac{\pi-\pi_0}{\pi_0}$ is the relative profit and loss.
- I_m is the quantity invested by tranche m, $1 \leq m \leq M$. And $D_m + 1 = \sum_{i=1}^{m-1} \frac{I_i}{\pi_0}$ is the relative cumulative quantity invested, $D_{M+1} = 0$.
- The default rate r^* is given by $L_m = I_m e^{r^*}$ where L_m is the undertaken future cashflows in one year of tranch m:

$$L_m = \begin{cases} I_m(1+R_{PL}) & \text{if} \quad R_{PL} > 0 \\ I_m & \text{if} \quad D_{m+1} < R_{PL} < 0 \\ I_m\left(\dfrac{R_{PL}-D_m}{D_{m+1}-D_m}\right) & \text{if} \quad D_m < R_{PL} < D_{m+1} \\ 0 & \text{if} \quad R_{PL} < D_m \end{cases} \quad (3)$$

- The recovery rate, f_m of tranche m, would be $\frac{R_{PL}-D_m}{D_{m+1}-D_m}$.
- r is the risk free rate.
- $s_m = r^* - r$ is the default spread.

Consider a portfolio π with underlying hedge funds $S_1, \ldots, S_n$, each with n_i positions. Its mark-to-market is given by:

$$\pi = \sum_{i=1}^{n} n_i S_i \quad (4)$$

and its profit and loss return is given by:

$$PL = \pi - \pi_0 = \sum_{i=1}^{n} \theta_i \left(\frac{S_i - S_i^0}{S_i^0} \right) = \sum_{i=1}^{n} n_i R_i \quad (5)$$

The tranche m will drop in default if:

$$\pi - \pi_0 = PL < \sum_{i=1}^{m-1} I_i - \pi_0$$

That is, if the mark-to-market of the portfolio at the end of the period is less than the invested quantity of the $m-1$ tranche of less seniority, then the m-th will default. We can express this inequality in terms of the relative profit and loss and the relative weights in each tranche:

$$R_{PL} = \frac{\pi - \pi_0}{\pi_0} < \sum_{i=1}^{m-1} \frac{I_i}{\pi_0} - 1 = D_m$$

Note that $D_1 = 0$. Therefore the probability of default from the m-th tranche, PD_m, equals:

$$PD_m = P(R_{PL} < D_{m+1}) \quad (6)$$

This equation relates the probability of default to the profit and loss distribution function of the underlying pool of hedge funds. Basically the one-year probability of default is related to the one-year VaR of the collateral portfolio. Under the assumption that the returns of the different hedge funds, R_i, follow a unidimensional CS process $CS\,(p, \lambda, r, \square_i^{\cdot,0}, \square_i^{\cdot,1}, \sigma_i^{\cdot,0}, \sigma_i^{\cdot,1})$, using a result presented in appendix, the profit and loss of the portfolio follows itself a unidimensional CS process with parameters $CS(p, \lambda, \Sigma_{i=1}^{d} \theta_i \square, \theta \cdot \square^0, \theta \cdot \square^1, \theta \cdot \sigma^0 \cdot \sigma^{0\prime} \cdot \theta^\prime, \theta \cdot \sigma^1 \cdot \sigma^{1\prime} \cdot \theta^\prime)$. Hence, in order to calculate the probabilities of default of the different tranche, we need only calculate the percentiles of a unidimensional CS process. Exhibit 3 presents a schematic picture of the model.

Moreover we can obtain the distribution function of the recovery rate of the tranche m, f_m, as:

$$P(f_m < x) = P\left(\frac{R_{PL} - D_m}{D_{m+1} - D_m} < x \mid D_m < R_{PL} < D_{m+1} \right) \quad (7)$$

$$= \frac{P(D_m < R_{PL} < x(D_{m+1} - D_m) + D_m)}{P(D_m < R_{PL} < D_{m+1})} \quad (8)$$

Subsequently, we can compute the spread of the m-th tranche as the discounted expected value of the one-year cashflow L_m under the risk-neutral probability Q:

$$I_m = L_m e^{-r_m^*}$$

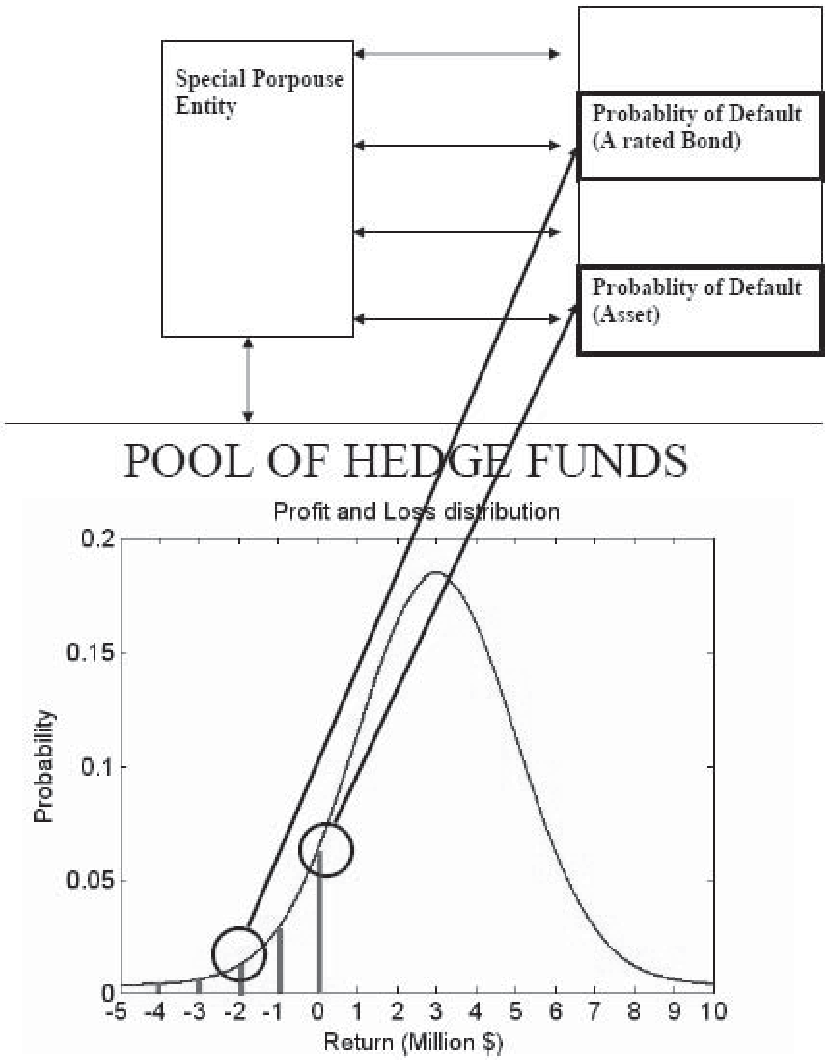
$$\begin{aligned} s_m &= r_m^* - r \\ &= \ln\left(E^Q \left[(1+R_{PL}) 1_{\{R_{PL}>0\}} + 1_{\{D_{m+1}<R_{PL}<0\}} \right.\right. \\ &\quad \left.\left. + (\frac{R_{PL} - D_m}{D_{m+1} - D_m} 1_{\{D_m < R_{PL} < D_{m+1}\}} \right] \right) \end{aligned} \quad (9)$$

We have to take into account that the yield spread between a corporate bond and an otherwise identical bond with no credit risk reflects the expected actuarial loss, or annual expected loss given default, plus a risk premia reflecting, for example, liquidity risk. It is possible to calculate the probability of default and the spread of tranche m; details are given in the appendix.

EXHIBIT 3

Schematic representation of the CFO valuation model

Probabilities of default are given by a percentile of the pool of hedge funds returns distribution. Recovery rates distribution and spread calculations are easy to obtain within this model.



A SAMPLE *CFO*

As explained before, CFOs are products which have not had too much impact among investors. This fact may be explained by the huge variability of the probability of default with respect to market conditions. To illustrate this feature, we analyze a CFO with the following characteristics: There is a total investment of $1 million dollars distributed within five tranches with weights 0.4976, 0.13, 0.0398, 0.065 and 0.2676 respectively. The tranche with the 26% is the equity tranche and the tranche with the 49% is the typically AAA rated tranche. These weights are in agreement with market conventions and were used in the first CFO created, Diversified Strategies CFO SA DSCFO1. With these assumptions, the limiting losses would be equal to $D_6 = 0$, $D_5 = -0.2676$, $D_4 = -0.3326$, $D_3 = -0.3724$ and $D_2 = -0.5024$.

The pool of Hedge Funds is chosen to be the following six Hedgefunds: BDC Offshore Fund Ltd (A), Kassirer Market Neutral, Mapleridge Fund LP, New

Ellington Overseas, Recap Inter and Styx International Ltd. We work with monthly data returns from Jan-2001 to Oct-2005. The investment in each hedge fund is equal to 1/6M$.

First we estimate the parameters of our model using the method proposed in the appendix. Notice that hedge funds report monthly returns instead of daily, leading to a lack of data. Our method is effective because it reduces the number of parameters, allowing for more reliable estimations and accurate intervals for the probabilities of default or for the spread calculations favoring the bid and offer pricing process.

In Exhibit 4 we present the estimations of the variances for all CTAs returns in the distressed and tranquil regimes and the mean under the historical measure. Variances are greater in distressed than in tranquil regimes. In Exhibit 5 we present the estimated correlations in the tranquil and distressed regime. One can observe the correlation breakdown phenomenon. Finally the proportion of tranquil months converges to 0.844 as t approaches infinity.

Using these estimations, we can calculate the parameters of the monthly relative profit and loss portfolio returns under the P-measure. Results are $\mu=0.101$, $\sigma^0 = 0.042$, $\sigma^1 = 0.089$, $p = 0.867$, $\lambda = 1.029$. In order to get the yearly returns distribution one needs to multiply monthly means and variances by twelve. With these results, using simulations and Equation (2), we are able to calculate the probabilities of default of the different tranches. In Exhibit 6 we can see that the equity tranche has a one-year probability of default of 19.5% and the most senior tranche a probability of default of 0.03%. Moreover we report the spread over the risk free rate, supposed to be 1%. Therefore a mezzanine tranche investor should be expecting a spread of 2.3% and the most senior tranche investor should be given a spread of 4.85 basis points. We do not report a spread value for the equity tranche because usually the yearly coupon on this tranche depends on the final portfolio value.

PARAMETER SENSITIVITIES

Up to now we have calculated the probabilities of default of the different tranches from the CFO according to market conditions during the period of Jan-2001 to Oct-2005. However, how would those probabilities have changed if market conditions had been slightly different? To answer that question we recalculate the probabilities of default, changing the probability p of a jump in Equation (1). In this way we measure the effect in default probabilities of a higher probability of being in a distressed regime. Therefore we valuate the CFO for a grid of probabilities in the interval from 0 to 1. Exhibit 7 shows that the probabilities of default spread over a substantial range when changing the probability of a distress month $1-p$ (market conditions). For example the mezzanine tranche probability of default could go from 2% to 9%. In Exhibit 8 we report the sensitivities of the spread yield to the market condition parameter p, which present a similar behavior to probabilities of default. In both cases a confidence interval for p could be attractive as a reliability measure. A good proxy for the confidence interval of the parameter p can be obtained using the binomial distribution, for example the 95% confidence interval for p ranges from 0.725 to 0.926, therefore the spread confidence interval is (1.9%, 3.8%). As a consequence investors could act with foresight when investing in this type of collateralized fund obligations.

CONCLUSION

A covariance switching multidimensional process was proposed and studied for the pricing of collateralized fund obligations (CFO). This process presents several desirable properties, such as closeness under linear transformation. This implies, in particular, that marginals and portfolios will belong to the CS family of processes. Another interesting property of this process is its simplicity for providing

EXHIBIT 4
Mean and Variances

	R_1	R_2	R_3	R_4	R_5	R_6
μ	0.012	0.0042	0.00520	0.0095	0.0073	0.00735
σ_t^2	0.0052	0.0039	0.0136	0.0078	0.0114	0.00313
σ_d^2	0.0186	0.001	0.0215	0.0127	0.021	0.00654

EXHIBIT 5

Lambdas in Factor analysis

	R_1	R_2	R_3	R_4	R_5	R_6
λ^0	0.564	0.088	0.082	–0.37	0.46	0.458
λ^1	0.862	0.0481	0.0154	–0.697	0.981	0.346

Tranquil Correlation

R_1	R_2	R_3	R_4	R_5	R_6
1	0.0497	0.046	–0.21	0.261	0.26
	1	0.0072	–0.033	0.041	0.040
		1	–0.031	0.038	0.037
			1	–0.172	–0.17
				1	0.21
					1

Distress Correlation

R_1	R_2	R_3	R_4	R_5	R_6
1	0.0415	0.132	–0.6	0.85	0.30
	1	0.0074	–0.033	0.047	0.017
		1	–0.108	0.15	0.053
			1	–0.69	–0.24
				1	0.34
					1

cumulative distribution probabilities in the unidimensional case, which enables us to compute portfolio cumulative probabilities by simulating a jump process. A method to estimate the parameters was also proposed. An empirical analysis shows a better than Gaussian fitting to a time series vector of hedgefunds; the large differences for the covariance matrices in tranquil and distress periods not only support the stylized facts described in the literature regarding leptokurtic unidimensional behaviors, but also the reported

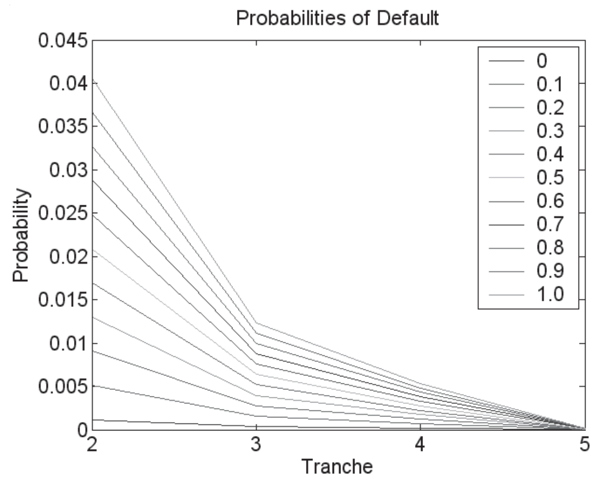
EXHIBIT 6

Estimated Probability of Default and Spread Yield for the Tranche

m	1	2	3	4	5
PD_m	19.56	3.84	2.6	0.65	0.03
S_m	4.85	2.3	2.1	1.14	0.1

EXHIBIT 7

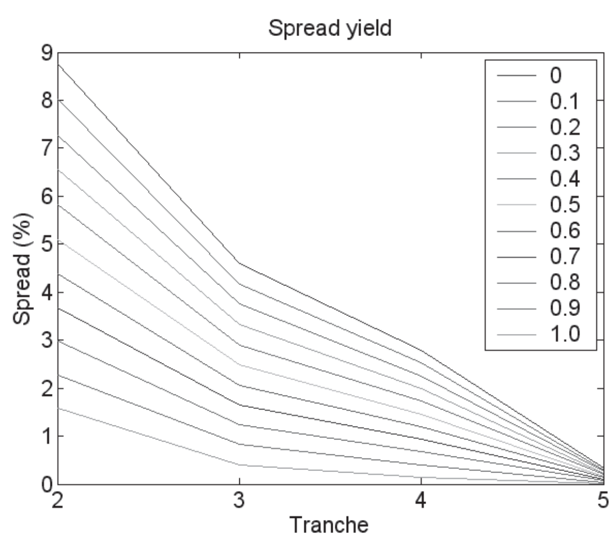
Sensitivities of Probabilities of Default to Market Conditions



correlation breakdown. We find that because of the lack of transparency characteristic of hedge funds, as well as the unavailability of data, confidence intervals in probabilities of default and credit spreads are high enough to burden CFO proliferation. For example, we find a 95% credit spread interval of 1.9%–3.8%.

EXHIBIT 8

Spread Yield Sensitivities to Market Conditions



APPENDIX

Covariance-Switching Process

A Covariance-Switching process is presented in this section.

We first define a jump process $J_t \in \{0, 1\}$. $P[\Delta J_t \in -1 \mid J_{t-}=1] = d(t)$, $P[\Delta J_t \in 1 \mid J_{t-} = 0] = u(t)$.

The law of a jump J_t is described by:

The intensity of jumps: $\lambda(t)dt$

The law of the jumps: $K(t, dy) = P[\Delta J_t \in dy \mid J_{t-}]$.

With previous notation, it follows:

$$K(t, dy) = P[\Delta J_t \in dy \mid J_{t-}]$$
$$= J_{t-}[(1-d(t))\delta_0 + d(t)\delta_{-1}](dy) + (1-J_{t-})[(1 - u(t))\delta_0 + u(t)\delta_1](dy)$$

where $\delta_x(dy)$ denotes the dirac delta. It is known that $J_t = J_0 + \int_0^t \int_E y\lambda(s)K(s, dy)ds + J_t^d$, where J_t^d is a local martingale (purely discontinuous martingale) such that $J_0^d = 0$ and $d\langle J_t^d, M\rangle = 0$ for any continuous local martingale M.

Remark: Notice that $\int_E y\lambda(s)K(s, dy) = J_{s-}(-d(s)) + (1-J_{s-})u(s)$. Then

$$J_t = J_0 + \int_0^t \left(J_{s-}(-\lambda(s)d(s)) + (1-J_{s-})\lambda(s)u(s)\right)ds + J_t^d$$

$$J_t = J_0 + \int_0^t \lambda(s)\,[u(s) - (d(s)+u(s))J_{s-}]\,ds + J_t^d$$

Moreover,

$$E[J_t \mid J_0] = J_0 + \int_0^t \lambda(s)\,(u(s) - (d(s)+u(s))E\,[J_{s-} \mid J_0])\,ds$$

Let us called $q(t) = E\,[J_t \mid J_0]$, then:

$$q(t) = \left[J_0 + \int_0^t \lambda(s)u(s)e^{\int_0^s \lambda(r)(u(r)+d(r))dr}ds\right] \cdot e^{-\int_0^t \lambda(s)(u(s)+d(s))ds} \quad (10)$$

Notice that $q(t)$ provides the probability of $J_t = 1$ given information up to $t = 0$. For simplicity, we assume the following parameters: $\lambda(t) = \lambda$, $u(t) = p$, $d(t) = 1 - p$, which leads to

$$q(t) = J_0 \cdot e^{-\lambda t} + p(1 - e^{-\lambda t}) \quad (11)$$

Definition: $X_i(t)$ follow a covariance switching process with parameters $(p, \lambda, \Box, \Box_i^0, \Box_i^1, \sigma_i^{\cdot,0}, \sigma_i^{\cdot,1})$ if the diffusion process can be represented as:

$$dX_i(t) = \Box_i(t)dt + \sigma_i(t)\cdot dW(t) \quad (12)$$
$$\Box_i(t) = \Box + J_t \cdot \Box_i^0 + (1-J_t)\cdot \Box_i^1$$
$$\sigma_i(t) = J_t \cdot \sigma_i^{\cdot,0} + (1-J_t)\cdot \sigma_i^{\cdot,1}$$
$$\sigma_i^{\cdot,k}(t) = (\sigma_i^{1,k}(t), \ldots, \sigma_i^{n,k}(t))$$

Where J_t is a jump process defined previously, $i = 1, \ldots n$, $j = 1, \ldots d$, W_t is a n-dimensional vector of independent Brownian motion processes, which are independent of J_t. Moreover, $\sigma_i^{j,k}$, $\Box_i^k$ are constants ($k = 0, 1$; $i = 1, \ldots, n$; $j = 1, \ldots, n$).

Property 1: If $P(t) = \Sigma_{i=1}^{d} a_i \cdot X_i(t)$, where X_i follows a CS process with parameters $(p, \lambda, \Box, \Box_i^0, \Box_i^1, \sigma_i^0, \sigma_i^1)$ then $P(t)$ follows a covariance-switching process with parameters $(p, \lambda, \Sigma_{i=1}^{d} a_i\Box, a\cdot\Box^0, a\cdot\Box^1, a\cdot\sigma^0\cdot\sigma^{0\prime}\cdot a', a\cdot\sigma^1\cdot\sigma^{1\prime}\cdot a')$.

Process for the underlying hedgefunds

$S_i(t)$ follow a covariance switching process with parameters $(p, \lambda, \Box_i, \sigma_i^{\cdot,0}, \sigma_i^{\cdot,1})$:

$$\frac{dS_i(t)}{S_i(t)} = \Box_i dt + \sigma_i(t)\cdot dW^P(t)$$
$$\sigma_i(t) = J_t \cdot \sigma_i^{\cdot,0} + (1-J_t)\cdot \sigma_i^{\cdot,1}$$
$$\sigma_i^{\cdot,k}(t) = (\sigma_i^{1,k}(t), \ldots, \sigma_i^{n,k}(t))$$

Where P-historical measure, J_t is the jump process defined previously, $i = 1, \ldots n$, $j = 1, \ldots d$, W_t is a n-dimensional vector of independent Brownian motion processes, which are independent of J_t. Moreover, $\sigma_i^{j,k}$ are constants ($k = 0, 1$; $i = 1, \ldots, n$; $j = 1, \ldots, n$).

We assume that the jump process does not change with a change of measure (see Merton [1974] for an explanation of the plausibility of this assumption), therefore if Q is the risk-free measure then,

$$\frac{dS_i(t)}{S_i(t)} = rdt + \sigma_i(t)\cdot dW^Q(t)$$

Property 2: Assumes $X_i(t) = \ln S_i(t)$ then, by Ito's lemma, $X_i(t) = rt - \frac{1}{2}\int_0^t \sigma_i(s)\cdot\sigma_i(s)'\,ds + \int_0^t \sigma_i(s)\cdot dW^Q(s)$.

Remark: The distribution of $X(t) = (X_1(t), \ldots, X_d(t))$ conditional on the history of J_s for $0 \le s \le t$, under the Q-measure, is multivariate Gaussian with mean and volatility as follows (t_j denotes the time of a jump, n-number of jumps, both known under the assumption):

$$\begin{aligned}\mu &= rt - \sum_{j=1}^{n}\left[\sum_{l=1}^{d}\left(\sigma_i^{l,k}\right)^2\cdot(t_j - t_{j-1})\right]\\ \sigma &= \sum_{j=1}^{n}\left[\sum_{l=1}^{d}\left(\sigma_i^{l,k}\right)\left(\sigma_j^{l,k}\right)\cdot(t_j - t_{j-1})\right]\\ k &= \sin^2\left(\frac{\pi}{2}j\right)\end{aligned}$$

For example:

$$\begin{cases} N_d(\mu^{(1)}, \Sigma^{(1)}) & \text{if} \quad J_s = 1, 0 \le s \le t \\ N_d(\mu^{(2)}, \Sigma^{(2)}) & \text{if} \quad J_s = 0, 0 \le s \le t \end{cases}$$

where the i component of $\mu^{(k)}$ ($k = 0, 1$) is $(r - \Sigma_{l=1}^{d}(\sigma_i^{l,k})^2)\cdot t$, while the i,j component of $\Sigma^{(k)}$ is $\Sigma_{l=1}^{d}(\sigma_i^{l,k})\,(\sigma_j^{l,k})\cdot t$.

These features suggest that a change in J_t can be seen as a change in the market conditions, leading to a change in trend not only for the volatility of the hedgefunds but also the correlation among them.

Property 3: Assumes $\Pi(t) = \Sigma_{i=1}^{d} a_i \cdot X_i(t)$, $\Sigma_{i=1}^{d} a_i = 1$, then, by Ito's lemma, $d\Pi(t) = [\mathrm{r} - \frac{1}{2}\Sigma_{i=1}^{d} a_i\sigma_i(t)\cdot\sigma_i(t)']\,dt + [\Sigma_{i=1}^{d} a_i\sigma_i(t)]\cdot dW^{Q}(t)$. Moreover, the distribution of $\Pi(t)$ conditional on observing the history of J_t is normal with the following mean and volatility:

$$\begin{aligned}\mu_\Pi(J) &= rt - \sum_{i=1}^{d} a_i\left[\sum_{j=1}^{d}\left\{\left(\sigma_i^{j,0}\right)^2 F_t + \left(\sigma_i^{j,1}\right)^2 (t - F_t)\right\}\right]\\ &= rt - \sum_{i=1}^{d} a_i\left(\sum_{j=1}^{d}\left(\sigma_i^{j,1}\right)^2 t\right)\\ &\quad + \left[\sum_{i=1}^{d} a_i\left(\sum_{j=1}^{d}\left(\left(\sigma_i^{j,0}\right)^2 - \left(\sigma_i^{j,1}\right)^2\right)\right)\right]F_t\\ &= A + B\cdot F_t\end{aligned} \tag{13}$$

$$\begin{aligned}\sigma_\Pi^2(J) &= \sum_{j=1}^{d}\left(F_t\cdot\left(\sum_{i=1}^{d} a_i\sigma_i^{j,0}\right)^2 + (t - F_t)\cdot\left(\sum_{i=1}^{d} a_i\sigma_i^{j,1}\right)^2\right)\\ &= \left(\sum_{j=1}^{d}\left(\sum_{i=1}^{d} a_i\sigma_i^{j,1}\right)^2 t\right) + \left[\sum_{j=1}^{d}\left(\left(\sum_{i=1}^{d} a_i\sigma_i^{j,0}\right)^2 - \left(\sum_{i=1}^{d} a_i\sigma_i^{j,1}\right)^2\right)\right]F_t\\ &= C + E\cdot F_t\end{aligned} \tag{14}$$

Where $F_t = \int_0^t J_s\,ds$.

Remark: The distribution of $\Pi(t)$ (given $\Pi(0)$) does not fit into a known family (unless conditional on J_t); this is not a drawback for pricing purposes, expectations can be computed by simulating the jump process J_t. i.e., for CFO pricing, we can proceed as follows:

$$\begin{aligned}P_0(\Pi(t) < D) &= E[1_{\{\Pi(t)<D\}} \mid \Pi(0)]\\ &= E[E[1_{\{\Pi(t)<D\}} \mid J, W(0)] \mid J(0)]\\ &= E\left[\Phi\left(\frac{D - \mu_\Pi(J)}{\sigma_\Pi(J)}\right)\middle| J(0)\right]\\ &= E[G(F_t) \mid J(0)]\end{aligned} \tag{15}$$

Where $G(F_t) = \Phi\left(\frac{D-\mu_\Pi(J)}{\sigma_\Pi(J)}\right) = \Phi\left(\frac{D-A+B\cdot F_t}{C+E\cdot F_t}\right)$, A, B, C, E were defined in property 3. Then we simulate paths of J (MonteCarlo) in order to compute $G(F_t)$.

Estimation

Hedge funds report returns on a monthly basis, leading to a lack of data for estimation purposes. Therefore it becomes necessary to consider multivariate models with as few parameters as possible. One way to achieve this is by describing the dependence structure of the underlying Multivariate Brownian processes using factor analysis.

The parameters of the model are estimated using the following two-step approach:

1. Search for the times where a jump has occurred. Here we start with an initial guess for distress months described before, then a sample point is removed if a test rejected the gaussianity hypothesis. We stopped as soon as gaussianity is accepted. Knowing the distress and the tranquil months allows us to compute the parameters of the jump process (p, λ) by fitting the theoretical probability path of a tranquil months to the empirical probability path: From Equation (8), the theoretical probability of a tranquil month at time t given $J_0 = 0$ is:

   $$q(t) = p\,(1 - e^{-\lambda t})$$

   The empirical probability would be: $\hat{q}(t) = \Sigma_{i=1}^{t}\frac{NT(t)}{t}$, where $NT(t)$ is the number of tranquil months by t. Therefore fiting the theoretical to the empirical we get:

   $$p = 0.867$$
   $$\lambda = 1.029$$

2. Factor analysis is used for estimating each of the covariance matrices for tranquil and distress events. We use

both sample data found in step 1 (returns from tranquil months and from distress months). Specifically, we propose a 1 factor model as follow:

$$c^0 \cdot dW^Q(t) = \lambda^0 \cdot dM^Q(t) + B^0 \cdot dZ^Q(t)$$
$$c^1 \cdot dW^Q(t) = \lambda^1 \cdot dM^Q(t) + B^1 \cdot dZ^Q(t)$$

where c^i is the correlation matrices under tranquil (i = 0) and distress (i = 1) conditions. λ^0, λ^1 are constant vectors and B^i are diagonal matrices with diagonal $\sqrt{1-(\lambda_j^{(1)})^2}$. This reduces the total number of parameters to be computed for the covariance matrices from $n^2 + n$ to $4n$. Means are computed using standard estimators.

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