

A Partial Differential Equation for Credit Derivatives Pricing.

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Abstract

Partial differential equations proved to be a fundamental tool as the derivatives market developed in the seventies. As markets continue into more sophisticated territory, such as credit trading, differential equations continue to play an important role, with the added qualifier that the equations that arise are now much more complex. This paper presents a broad overview into the credit markets, and discusses a partial differential equation that arises in that context.

Key Words: Credit Derivatives, Partial differential equations, minimum Stochastic Processes.

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1 Introduction

Financial activity is an affair in risk transfer. Stocks and bonds, the financial instruments of the XIX century, are designed to allow investors to participate in commercial enterprises; stock holders assume market risk, i.e., the risk that the firm does not meet profitability expectations; bond investors are not exposed to that market risk, and only assume default risk, i.e., the risk that the issuing entity cannot meet its financial obligations. This is also called *credit risk*, and losses can also occur without the company defaulting: a mere credit downgrade will lead to a decrease in the market value of the bond, and hence a loss, realized or not.

In the latter part of the XX century, market risk was traded massively through the derivatives market. Investors could buy price protection related to stocks, currencies, interest rates or commodities by purchasing options or other derivatives; some are standard, others are tailor-made and labelled "over-the-counter".

But as opposed to traditional insurance premiums, financial insurance is not based merely on probabilistic considerations. To see why, imagine the following a very simple hypothetical situation.

There is an asset (a stock, a home, a currency, etc.) trading today at \$1, which can only be worth \$2 or \$0.50 next year, with equal probability; interest rates are 0%, i.e., borrowing is free. Consider also an investor which may need to buy this asset next year and is therefore concerned with increase in value; for that reason decides to buy insurance in the following form: if the asset raises to \$2, then the insurance policy will pay \$1. If the asset drops in price however, the policy pays nothing. One would be tempted to price this insurance policy with a premium obtained through past price movements of that asset, and it would seem that \$0.50 is the price that makes sense.

If the investor paid \$0.50, then the seller of the policy could do the following: borrows an additional \$0.10, and buys 60% of the stock. If the stock raises in value, after paying the \$1 and returning the loan they would make a profit of \$0.10. If, however, the stock drops in price, they will make a net profit of \$0.20, as the policy pays nothing and they only need to return the loan. In other words, \$0.50 is *too much*, as the issuer of the option will always make a profit: this phenomenon is called arbitrage, and it is a fundamental assumption for pricing theories that arbitrage should not exist. A simple calculation will show that the no-arbitrage price is exactly \$1/3.(see fig 1).

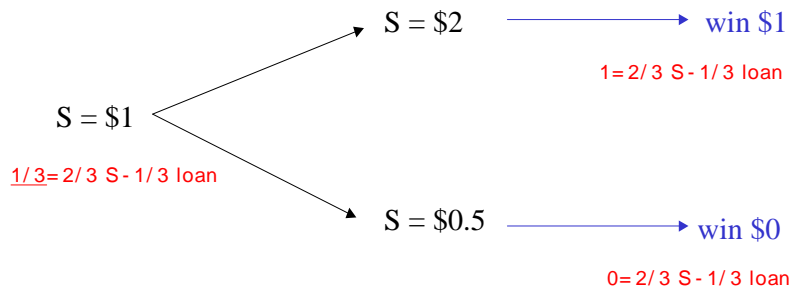


Figure 1: Pricing Fundamentals

This simple example (a "call option") is the basis of the no-arbitrage pricing theory (J. Hull and A. White 2000), and we can quickly learn a few things from it. First, the price of a contract that depends on market moves may be replicated with buy/sell strategies, which mimic the contract pay-out but can be carried out with fixed, pre-determined costs. Second, there is a probability of events which is implied by their price, and is perhaps independent of historical events. In our example above, the implied probability of an up-move is 33%, and the probability of a down-move is 66%, because with those probabilities we can price the contract taking simple expectations. However, a more profound revision of the previous example will convince the reader with a background in diffusion processes that, if one takes the simple one-step example into a continuum of infinitesimal time/price increments, one ends with Brownian motion and associated Kolmogorov forward operator: the heat equation. One will also have a diffusion process for the asset or stock, and an associated diffusion process implied by market prices.

Black, Scholes and Merton, in 1973, derived the analog of the heat equation and Brownian motion for the case of an option with an underlying stock price that follows the Ito process given by

$$dS = \mu S dt + \sigma S dW^P,$$

where here S denotes the stock price, μ denotes the drift, σ is the volatility, and dW^P are infinitesimal Brownian increments. An option on a stock is as a contract that will pay a future value at expiration $f_0(S)$, which depends on the value of the underlying stock S at expiration time T . Note the similarity with our simple example above, the main difference being that in our case now the stock trades continuously and we could therefore replicate our option by trading the stock continuously. In this case, the Black-Scholes-Merton theory shows that the price of the option contract is obtained by solving the following backward parabolic PDE (for $t < T$):

$$\begin{cases} \frac{\partial f}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 f}{\partial S^2} + rS \frac{\partial f}{\partial S} - rf = 0 \\ f(S, T) = f_0 \end{cases}$$

At first sight, this expression has two counterintuitive features: the absence of μ and the presence of the interest rate r in the PDE. A moment's reflection however, will convince us that this is not entirely surprising: after all, in our example in Figure 1 we already saw that the price of that option was independent of the probabilities of up and down moves of the stock, and it will only depend on the cost of borrowing. This was forced on us by our no-arbitrage assumption. In more general terms, it turns out that option pricing can be established by taking expectations with respect to a "risk neutral" measure Q , which is perhaps different from the historical measure P . In our particular case, this implies that the solution to the PDE is given by

$$f(S, t) = \frac{1}{\sqrt{2\pi(T-t)}\sigma} \int_0^\infty f(u) \exp \left\{ -\frac{\left(\ln \left(\frac{u}{S(t)} \right) - \left(r - \frac{\sigma^2}{2} \right) (T-t) \right)^2}{2(T-t)\sigma^2} \right\} du$$

which is easily checked. More generally, we have that

$$\begin{aligned} f(S, t) &= E[f(S, T) | S(t)] \\ dS &= rS dt + \sigma S dW^Q, \end{aligned}$$

From this perspective, pricing becomes equivalent to finding risk neutral probabilities for a certain new measure Q (the risk neutral measure) and their pay-off expectations, and the PDE above is nothing but the Feynmann-Kac formula for this expectation.

The Black-Scholes-Merton theory also shows that one can replicate the option payoff by continuously trading the stock so that we always own $\partial_S f$ units of it.

This signified a tremendous revolution, that won Black and Scholes the Nobel prize for Economics in 1997, as it not only established a pricing mechanism for the booming options and derivative markets, but because -in highly idealized conditions, of course- it established certainty where there was risk: derivatives could be replicated by buy/sell strategies with predetermined costs.

Their discovery revolutionized market risk perspectives. But Merton, who had re-derived their

pricing formalism using stochastic control theory, used this advance to start the modern theory of credit risk. His viewpoint, which we present below, was just as revolutionary.

Merton viewed a firm as shareholders and bond-holders. Bond-holders lent money to the firm, and the firm promised to pay back the loan, with interest. Shareholders own the value of the assets of the firm, minus the value of the debt (or liabilities); but firms have limited liability, which means that if the value of the assets falls below the value of the liabilities, in Merton's view, the firm defaults, shareholders owe nothing and the bondholders use the remaining value of the assets to recover a portion of their loan. In other words, the shareholders own a call option on the value of the assets of the firm, with a strike price given by the value of the liabilities at the given maturity time of the loan. The timing of his theory, which dates back to 1974, was perfect as the theory of option pricing had just been developed one year earlier, and this opened the ground for credit risk pricing and credit risk derivatives.

Strictly speaking, Merton approach assumes that the liabilities of a firm (its debt) expire at a certain time, and default could occur only at that time. Black and Cox (1976) conceptually refined Merton's proposal by allowing defaults to occur at anytime within the life of the option, creating the "first passage default models". The reason for this modification is that, according to Merton's model, the firm value could dwindle to nearly nothing without triggering a default until much later; all that matters was its level at debt maturity and this is clearly not in the interest of the bond holders. Bond indenture provisions therefore often include safety covenants providing the bond investors with the right to reorganize or foreclose on the firm if the asset value hits some lower threshold for the first time. This threshold could be chosen as the firm's liabilities.

But the largest event in the credit market still had to wait until 1998, when the default of Russia and the menace of the impeachment of President Clinton over the Monica Lewinsky affair threw financial markets into disarray; the Russian default, and worries about the political stability of the United States created a *credit crunch* as bond investors fled from corporate debt for the more secure treasury bill market, introducing credit spread dislocations of historical proportions. This situation culminated with the collapse of Long Term Capital Management, a multi-billion dollar hedge fund that, anecdotally, had lured Scholes and Merton to their board of directors.

The result of these massive historical events was the explosion of the credit market. In it, financial players seek to buy and sell credit risk, either for insurance and protection in the case of default or bankruptcy of their counter-parties, or to take risk exposure which are considered either

cheap or advantageous, and therefore earn above average returns. The financial instruments which are used in the credit market are numerous. In this paper, we shall focus on two closely related ones: credit default swaps (CDS), and collateralized debt obligations (CDO).

A credit default swap (CDS, see fig 2) is a contract that provides insurance against the risk of a default by particular company (known as the reference entity). The buyer of the insurance obtains the right to sell a particular bond issued by the company for its par value when a credit event occurs. The bond is known as the reference obligation and the total par value of the bond that can be sold is known as the swap's notional principal. The buyer of the CDS makes periodic payments to the seller until the end of the life of the CDS or until a credit event occurs. A credit event usually requires a final accrual payment by the buyer.

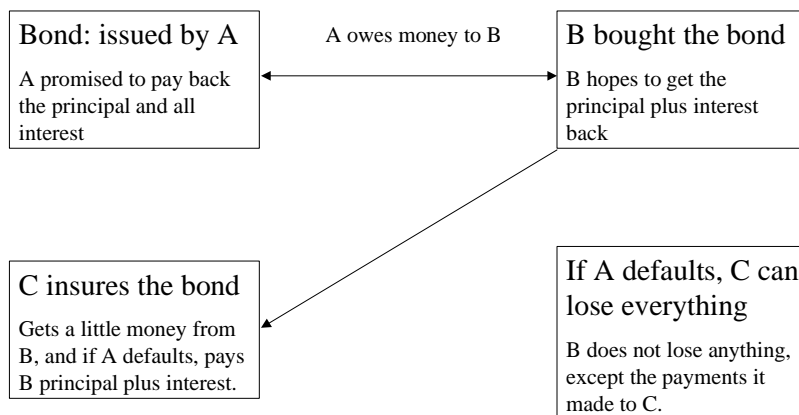


Figure 2: Credit Derivatives

A collateralized debt obligation (CDO) provides a way of creating securities with different risk characteristics from a portfolio of debt instruments. A general example would be when M types of securities are created from a portfolio of N bonds. The first tranche of securities has p_1 of the total bond principal and absorbs all credit losses from the portfolio during the life of the CDO until they have reached p_1 of the total bond principal. The second tranche has p_2 of the principal and absorbs all losses during the life of the CDO in excess of p_1 of the principal up to a maximum of $p_1 + p_2$ of the principal. The last tranche has p_M of the principal absorbs all losses in excess of $\sum_{i=1}^{M-1} p_i$ of the principal. The reason these instruments exist is that banks with large loan books, can use CDO's to effectively slice the default risk in those portfolios with credit-linked securities (the different tranches) and sell them to investors in packets which exhibit very different risk profiles: from the highly risk of the top -mezzanine- tranche (which will earn a higher fee spread), to

the very secure last tranche, which will earn perhaps a minimal fee.

The building blocks of a CDO are the n^{th} -to-default credit default swaps. An n^{th} to default credit default swap (nCDS) is similar to a regular CDS. The buyer of protection pays a specified rate on a specified notional principal until the n^{th} default occurs among a specified set of reference entities or until the end of the contract's life. The payments are usually made quarterly. If the n^{th} default occurs before the contract maturity, the buyer of protection can present bonds issued by the defaulting entity to the seller of protection in exchange for the face value of the bonds. Alternatively, the contract may call for a cash payment equal to the difference between the post-default bond value and the face value. (see fig 3).

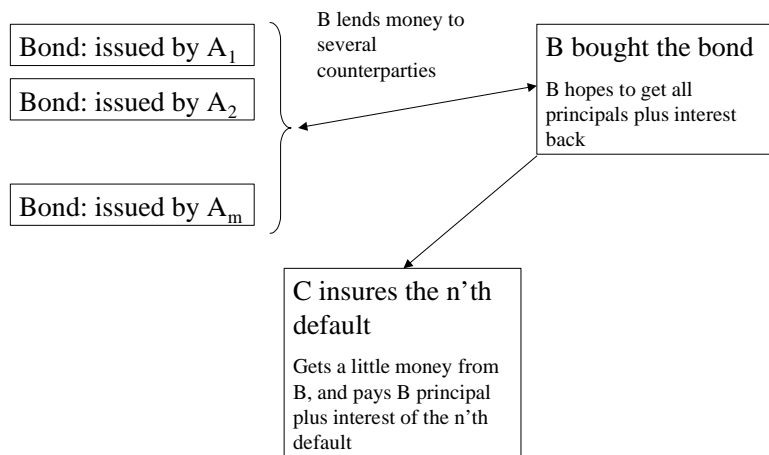


Figure 3: n^{th} to default Swap

The valuation of such structures are based on computing the probability distribution of the event " m^{th} default". This is technically difficult because it requires one to handle the multivariate distribution of defaults, and generally most credit models fail to reliably capture multiple defaults. There are basically two procedures for evaluating these basket derivatives, multifactor copula models similar to that used by researchers such as Li (2000) and Laurent and Gregory (2003) and Hull and White (2004) and intensity models, see Duffie-Garland 2001.

In this paper we develop a partial differential equation (PDE) procedure for valuing both an n^{th} to default CDS and a CDO. We work within the structural framework, where the default event is associated to whether the minimum value of an stochastic processes (firm's asset value) have reached a benchmark, usually the firm's liabilities (see Merton 1973, Black-Cox 1976). We

show that the multivariate density/distribution of a vector of Brownian motions and Brownian minimums (it can be easily extended to maximums) for the case of more than two underlying components is the solution of a PDE with absorbing and boundary conditions (a Fokker-Planck equation).

The paper is organized as follows. In section 2 we address the pricing of credit derivatives, section 2.1 provides the set up of the economic and mathematical assumptions and the joint density/distribution functions of the minimums values of n correlated Brownian motions. Next, in section 2.2 we do some applications, an n^{th} to default CDS in section 2.2.1, while a CDO is priced in 2.2.2.

2 Pricing Credit Derivatives.

In a standard credit framework, there are n reference firms, C_1, \dots, C_n . The protection buyer A and the protection seller B. A pays a regular fee (initial fee) to B until a pre-specified event occur or maturity. B agrees on paying the total default. One of the method for pricing credit derivatives is the structural approach.

In the structural approach, one makes explicit assumptions about the dynamics of a firm's assets, its capital structure, as well as its debt and shareholders. The philosophy of the structural approach, which goes back to Black & Scholes (1973) and Merton (1974), is to consider corporate liabilities as contingent claims on the assets of the firm. Here the firm's market value (the total market value of the firm's assets, denoted V_T) is the fundamental state variable. The most popular structure in practice is based on Merton's (1974) framework, where default can only occur at maturity (whenever $V_T < D$). Recognizing that a firm may default well before the maturity of the debt, one may alternatively assume that the firm goes bankrupt when the value of its assets falls below some lower threshold (that is, when $\min_{0 \leq s \leq t} V(s) < D$, see Black-Cox (1976).

The most prominent and challenging feature of a credit instruments is the dependency structure of defaults. This is why most recent efforts have been directed to the capture of these dependency structures. Structural framework's handling of this problem rest on the finding of the multivariate distribution of first-passage-time. This distribution can be inferred from the multivariate distrib-

ution of a vector of Brownian minimums (see Zhou 2001 and Giesecke 2003).

We know, based on the Black-Scholes framework, that the pricing of derivatives can be reduced to finding the expected value of the derivative's payoff under a "free-arbitrage" equivalent measure.

2.1 Assumptions and Results

We first present the very basic assumptions required in our work.

- Interest rate, r is constant.
- The value of the assets, $V_i(t)$, follows an Ito process with constant drift r and volatility $\sigma_i^2(t)$ (an adapted process such that $\int_0^t (\sigma_i(s))^2 ds < \infty$ a.s) under the risk neutral measure:

$$dV_i = rV_i dt + \sigma_i(t)V_i dW_i(t).$$

- Firm i defaults as soon as its asset value V_i reaches the liabilities, denoted as D_i . This is the definition of default within the structural framework (see Merton 1974, Black-Cox 1976, Giesecke 2003).

Let us take $X(t) = \ln V(t)$ as the n -dimensional Brownian motion vector with drift $\mu = (\mu_1, \dots, \mu_n)$ (where $\mu_i(t) = r - \frac{1}{2}\sigma_i^2(t)$) and covariance matrix $\Sigma = (\sigma_{i,j}(t)) = (\rho_{ij}\sigma_i(t)\sigma_j(t))$,

$$X(t) = \mu(t) \cdot t + \text{diag}(\Sigma(t)) \cdot W(t) \quad , \quad t \geq 0 \quad , \quad (1)$$

where $W(t)$ is a correlated Brownian vector such that $E[dW_i dW_j] = \rho_{ij} dt \quad \forall i \neq j = 1, \dots, n$.

The two most important variables in our framework are the running minimum of the process $X(t)$ and the time of default t_i , they are defined as follows:

The running minimum is defined as:

$$\underline{X}_i(t) = \min_{0 \leq s \leq t} X_i(s),$$

While the time of default for firm i is defined using the running minimum as:

$$t_i = \inf \{t | \underline{V}_i(t) \leq D_i\} = \inf \{t | \underline{X}_i(t) \leq \ln D_i\} \quad (2)$$

When handling several companies simulatenously, those previous definitions are not enough. The time of default of a given company is a marginal piece of information which fails to hint at other companies' movements. Therefore it is useful to create variables aimed at describing joint defaults scenarios. In this regard the variables τ_j :time of j -defaults is defined next:

The time of j -defaults is defined as: $\tau_j = t_{i_j}$ where $t_{i_1} \leq \dots \leq t_{i_n}$.

Having defined all these key variables, we next show that their probability distribution can be obtained from the joint probabilities of running minimums. The reason why we emphasize this relationship is because the joint probability of running minimums (as well as running maximum in this regard) is known. It is actually the solution of a partial differential equation.

Denote $\pi_t(1)$ for the probability of exactly one default before t . Then

$$\pi_t(1) = P(\tau_1 \leq t, \tau_2 > t) = \sum_{i=1}^n P(\underline{V}_1(t) > D_1, \dots, \underline{V}_i(t) \leq D_i, \dots, \underline{V}_n(t) > D_n). \quad (3)$$

In general, the probability of exactly j defaults before t could be found as:

$$\pi_t(j) = P(\tau_j \leq t, \tau_{j+1} > t) \quad (4)$$

$$= \sum_{i_1 \neq i_j = 1} P(\underline{V}_1(t) > D_1, \dots, \underline{V}_{i_1}(t) \leq D_{i_1}, \dots, \underline{V}_{i_j}(t) \leq D_{i_j}, \dots, \underline{V}_n(t) > D_n), \quad (5)$$

where the summation is taken over the $\frac{k!}{j!(k-j)!}$ different ways in which the \underline{V}_i can be chosen.

The probability of at least one default before t , $P(1, t)$, would then be:

$$\begin{aligned} P(1, t) &= P(\tau_1 \leq t) \\ &= [1 - P(\underline{V}_1(t) > D_1, \dots, \underline{V}_i(t) > D_i, \dots, \underline{V}_n(t) > D_n)]. \end{aligned} \quad (6)$$

Similarly, the probability of at least j defaults before t would be:

$$P(j, t) = P(\tau_j \leq t) = \sum_{k=j}^N \pi_t(k). \quad (7)$$

Let us also denote $f_j(\cdot)$ as the density of τ_j . Then

$$f_j(t) = \frac{\partial P(j, t)}{\partial t}. \quad (8)$$

Notice that in order to compute those previously defined probabilities, we need the multivariate distribution of the vector of minimums of X . This is tackled in the next result.

Theorem 1: Let us denote:

$$\begin{aligned} &P(X_1(t) \in dx_1, \dots, X_n(t) \in dx_n, \underline{X}_1(t) > m_1, \dots, \underline{X}_n(t) > m_n) \\ &= p(x_1, \dots, x_n, t, m_1, \dots, m_n, \mu, \Sigma) dx_1 \dots dx_n, \end{aligned} \quad (9)$$

where $m_i = \ln(D_i)$, $p(x_1, \dots, x_n, t, m_1, \dots, m_n, \mu, \Sigma)$ is the joint distribution/density function of the minimums and endpoints of $X(t)$, $x_i > m_i$, $m_i \leq 0$; $\forall i$. Then p satisfies the following partial differential equation (Fokker-Planck equation):

$$\begin{cases} \frac{\partial p}{\partial t} = - \sum_{i=1}^n \mu_i(t) \cdot \frac{\partial p}{\partial x_i} + \sum_{i,j=1}^n \frac{\sigma_{ij}(t)}{2} \cdot \frac{\partial^2 p}{\partial x_i \partial x_j} \\ p(x, t=0) = \prod_{i=1}^n \delta(x_i) \\ p(x_1, \dots, x_i = m_i, \dots, x_n, t) = 0, \quad i = 1, \dots, n \\ x_i > m_i, m_i \leq 0, \quad i = 1, \dots, n \end{cases}$$

Proof. What follows is a sketch of the proof of theorem 1. Suppose that $X(t)$ follows the process described in equation 1, consider any twice differential function $p(X, t)$ that is a conditional expectation of some function of X at a later date T , $g(X(T))$:

$$p(X, t) = E [g(X(T)) | X(t) = x] \quad (10)$$

By Ito's lemma, we can compute the stochastic process for f . Moreover, note that $p(X, T) = g(X(T))$, therefore f is a martingale. This implies that the drift of the process for p has to be zero, leading to the equation:

$$\frac{\partial p}{\partial t} = - \sum_{i=1}^n \mu_i(t) \cdot \frac{\partial p}{\partial x_i} + \sum_{i,j=1}^n \frac{\sigma_{ij}(t)}{2} \cdot \frac{\partial^2 p}{\partial x_i \partial x_j}$$

with boundary condition $p(X, T) = g(X(T))$.

Selecting a convenient function g is key for getting a meaning for the function p , for example if $g(v, \sigma_i) = \exp \{i\theta_1 v_1 + i\theta_2 v_2\}$ then the solution is the characteristic function for X . In our case we take $g(X) = \delta(X - X_0)$ therefore the solution $p(X, t)$ is the conditional probability density at time t that $X(T) = X_0$.

To see the absorbing boundary condition notice that we ask for the probability density $p(x_1, \dots, x_n, t)$ for the stochastic vector $X(t)$, starting at $t = 0$ with $X(0) = 0$ to reach $x = (x_1, \dots, x_n)$ at time t . We do the following in order to make the path of $X_i(t)$ always greater than m_i , ($\underline{X}_i(t) > m_i, \forall i$): if the stochastic component X_i reaches m_i for the first time we no longer count these vector realizations (this would assure $\underline{X}_i(t) > m_i$). Therefore $p(x, t, \mu, \Sigma)$ must be zero for $X_i \leq m_i, \forall i$. The prescription that the processes are no longer counted if they passed a boundary is substituted by an absorbing n dimensional wall ($p(x_1, \dots, x_i = m_i, \dots, x_n, t) = 0, i = 1, \dots, n$)³. For the domain $m \leq X$, $p(x, t, m, \mu, \Sigma)$ must satisfy the previous Fokker-Planck equation ⁴ (see Risken 1989). □

Remark 1. Note that integrating p with respect to x leads to the distribution function of the minimum:

$$\begin{aligned} P(\underline{X}_1(t) > m_1, \dots, \underline{X}_n(t) > m_n) \\ = \iint p(x_1, \dots, x_n, t, m_1, \dots, m_n, \mu, \Sigma) dx_1 \dots dx_n, \end{aligned}$$

which is the key element for computing the quantities $\pi_t(j)$, $P(j, t)$ and $f_j(t)$ on equations 5, 7 and 8 respectively.

³For a given t , p gives the probability that the path lies within the domain, so not only $x_i(t) \geq m_i$ but also $\underline{X}_i(t) \geq m_i$.

⁴It describes the time evolution of the probability density function of position and velocity of a particle, it is also known as forward Kolmogorov equation.

In a one dimensional setting, assuming constant drift and volatility $P(X_1(t) \in dx_1, \underline{X}_1(t) > m_1) = p(x_1, t, m_1, \mu_1, \sigma_1^2)dx_1$, the PDE becomes:

$$\begin{cases} \frac{\partial p}{\partial t} = -\mu_1 \cdot \frac{\partial p}{\partial x_1} + \frac{\sigma_1^2}{2} \cdot \frac{\partial^2 p}{\partial x_1^2} \\ p(x_1, t = 0) = \delta(x_1) \\ p(x_1 = m_1, t) = 0 \\ x_1 > m_1, m_1 \leq 0 \end{cases}$$

A solution to this PDE can be obtained combining transformations and the reflection principle. First removing the drift term by a transformation $q(x_1, t) = p(x_1, t)e^{ax_1+bt}$ for convenient values of a and b . Notice that $q(x_1, t)$ can be seen as the solution to $q(x_1, t)dx_1 = P(X_1(t) \in dx_1, \underline{X}_1(t) > m_1)$ where $X_1(t)$ is a plain brownian process without drift. Then we find $q(x_1, t)$ using the reflection principle.

The reflection principle is a symmetry principle that states that $P(W_1(t) < w_1, \underline{W}_1(t) < m_1) = P(W_1(t) < 2m_1 - w_1)$, or,

$$P(W_1(t) < w_1, \underline{W}_1(t) > m_1) = P(W_1(t) < w_1) - P(W_1(t) < 2m_1 - w_1).$$

It then follows that:

$$q(x_1, t) = \frac{\partial P(X_1(t) < x_1, \underline{X}_1(t) > m_1)}{\partial x_1}$$

Substituting $q(x_1, t)$ back into the equation $p(x_1, t) = q(x_1, t)e^{-ax_1-bt}$ leads to:

$$p(x_1, t, m_1, \mu, \sigma^2) = \frac{1}{\sigma_1 t} \phi\left(\frac{x_1 - \mu_1 t}{\sigma_1 \sqrt{t}}\right) \left(1 - \exp\left\{\frac{-4m_1^2 + 4m_1 x_1}{2\sigma_1^2 t}\right\}\right)$$

Notice that integrating with respect to x_1 leads to the distribution function of the minimum, which is called the inverse Gaussian distribution

$$p(\underline{X}_1(t) \geq m_1) = \Phi\left(\frac{m_1 - \mu_1 t}{\sigma_1 \sqrt{t}}\right) - \exp\left\{\frac{2m_1 \mu_1}{\sigma_1^2}\right\} \Phi\left(\frac{-m_1 - \mu_1 t}{\sigma_1 \sqrt{t}}\right). \quad (11)$$

(see for example Giesecke 2003).

He, Keastead and Rebholz 1998 provided the joint density for the case of two Brownian processes $(X_1, \underline{X}_1, X_2, \underline{X}_2)$ assuming constant drift and volatility. This can also be seen as a particular case of this framework, Remark 1 taking $k = 2, n = 2$. They pointed out that the joint distribution/density $p(x_1, x_2, t, m_1, m_2, \mu, \Sigma)$ for $x_1 \geq m_1$, $x_2 \geq m_2$ where $m_i \leq 0$ must satisfy the following Partial differential equation:

$$\begin{cases} \frac{\partial p}{\partial t} = - \sum_{i=1}^2 \mu_i \cdot \frac{\partial p}{\partial x_i} + \sum_{i,j=1}^2 \frac{\sigma_{ij}}{2} \cdot \frac{\partial^2 p}{\partial x_i \partial x_j} \\ p(x, t = 0) = \prod_{i=1}^2 \delta(x_i) \\ p(x_i = m_i, t) = 0, \quad i = 1, 2 \\ x_i > m_i, m_i \leq 0, \quad i = 1, 2 \end{cases}$$

The solution of this PDE is:

$$p(x_1, x_2, t, m_1, m_2, \mu, \Sigma) = \frac{e^{a_1 \cdot x_1 + a_2 \cdot x_2 + b \cdot t}}{\sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \cdot h(x_1, x_2, t, m_1, m_2, \mu, \Sigma), \quad (12)$$

where: a and b are the solution of the following system of equations:

$$0 = \mu + \sigma_{i,j} \cdot a_j \quad (13)$$

$$b = \sum_{i,j=1}^2 \sigma_{ij} \cdot a_i \cdot a_j + \sum_{i=1}^2 \mu_i \cdot a_i, \quad (14)$$

while $h =$:

$$\frac{2}{\beta t} \sum_{l \in \mathbf{N}} e^{-\frac{r^2 + r_0^2}{2t}} \cdot \sin \frac{l \cdot \pi \cdot \theta_0}{\beta} \cdot \sin \frac{l \cdot \pi \cdot \theta}{\beta} \cdot I_{(l\pi/\beta)}\left(\frac{r \cdot r_0}{t}\right). \quad (15)$$

Here:

$$\begin{aligned} z_1 &= \frac{1}{\sqrt{1-\rho^2}} \left(\frac{x_1 - m_1}{\sigma_1} - \rho \frac{x_2 - m_2}{\sigma_2} \right) & z_2 &= \left(\frac{x_2 - m_2}{\sigma_2} \right) \\ \tan \theta &= \frac{z_2}{z_1}, \theta \in [0, \pi] & \tan \beta &= -\frac{\sqrt{1-\rho^2}}{\rho}, \beta \in [0, \pi] \\ z_{10} &= \frac{1}{\sqrt{1-\rho^2}} \left(\frac{-m_1}{\sigma_1} + \rho \frac{m_2}{\sigma_2} \right) & z_{20} &= -\frac{m_2}{\sigma_2} \\ r &= \sqrt{z_1^2 + z_2^2} & r_0 &= \sqrt{z_{10}^2 + z_{20}^2} \\ \tan \theta_0 &= \frac{z_{20}}{z_{10}}, \theta_0 \in [0, \beta] & \rho &= \frac{\sigma_{i,j}}{\sigma_i \sigma_j} \end{aligned}$$

$I_n(x)$ is the Bessel function of the second kind, fundamental solution of the Bessel equation:

$$x^2 \frac{d^2 Y}{dx^2} + x \frac{dY}{dx} + (x^2 - n^2) Y = 0$$

Integrating with respect to (x_1, x_2) leads to the joint distribution function of the minimums.

The main reason why this PDE could be solved, in closed form, for dimension two is the existence of an orthogonal coordinate system (polar coordinates) that allows for the removal of the mixing derivative term in the PDE equation while keeping a separable PDE equation with orthogonal boundary conditions. The procedure for finding the solution was based on three steps, first removing the drift term (using a similar transformation to that in the unidimensional case), then a change to polar coordinates removes the mixing derivative while keeping the boundary conditions orthogonal. The remaining PDE after those transformations is a Laplace equation which was solved explicitly using separation of variables.

A closed form solution for the n dimensional case, assuming constant drift and volatility, has not been found yet. The most important reason for this failure could be traced to the non existence of an orthogonal coordinate system (within systems of degree two) capable of removing the mixing derivative while keeping the boundary conditions orthogonal. For example in the three dimensional case, it is known that there are 11 orthogonal coordinate systems in which the Laplace equation can be solved by separation of variables, but none of them fulfills both objectives: removal of mixing derivatives while orthogonalizing the boundary conditions (see Moon and Spencer 1988).

Remark 2. *This joint distribution/density can be manipulated in order to obtain the joint distribution/density of a subset of endpoints and minimums; for example $P(X_1(t) \in dx_1, \dots, X_n(t) \in dx_n, \underline{X}_1(t) \geq m_1, \dots, \underline{X}_k(t) \geq m_k), P(X_{k+1}(t) \in dx_{k+1}, \dots, X_n(t) \in dx_n, \underline{X}_1(t) \in dx_1, \dots, \underline{X}_k(t) \in dx_k)$ by integration.*

Remark 3. *The previously defined quantities, $\pi_t(j)$ and $P(j, t) \forall j$, can be obtained by using theorem 1. Specifically we can imply the density $g(t, m_1, \dots, m_n)$ of the multivariate distribution of $(\underline{V}_1, \dots, \underline{V}_n)$.*

2.2 Applications

2.2.1 n^{th} to default CDS.

As explained previously, an n^{th} to default credit default swap (CDS) is similar to a regular CDS.

The buyer of protection:

- **Pays** a specified rate on a specified notional principal until the n^{th} default occurs among a specified set of N reference entities or until the end of the contract's life. The payments are usually made quarterly (we assume only an initial payment).
- **Receives** If the n^{th} default occurs before the contract maturity, the buyer of protection can present bonds issued by the defaulting entity to the seller of protection in exchange for the face value of the bonds.

Assumptions

1. Principals associated with all the underlying entities are the same, $L = 1$ and the expected recovery rates are the same, $R(\tau_j)$.
2. In the event of j^{th} defaults, the seller pays the notional principal times $(1 - R(\tau_j))$.
3. For the sake of simplicity we assume only one payment (from the buyer) at the beginning of the contract. (Usually the buyer of protection makes quarterly payments in arrears at a specified rate until the j^{th} default occurs. It is also common to define it so that there is a payoff for the first j defaults (rather than just for the j default); also, sometimes the rate of payment reduces as defaults occur.)

Proposition 1. *The present value (t) of a j^{th} to default CDS can be computed, based on Black-Scholes arguments, as follows:*

$$\begin{aligned} E_t^Q \left[(1 - R(\tau_j)) \cdot e^{r(\tau_j - t)} \cdot 1_{\{\tau_j < T\}} \right] \\ = \int_t^T (1 - R(s)) \cdot e^{r(s-t)} \cdot f_j(s) ds. \end{aligned} \tag{16}$$

Notice that the density $f_j(t)$ can be computed from the solution to the Fokker-Planck equation (see Theorem 1, equation (8) and Remark 1).

2.2.2 Collateralized Debt Obligation

We keep the same assumptions as in a CDS, i.e. the expected recovery rates (R) and principals (L) associated with all the underlying names are the same.

A CDO is a way of creating M securities with different risk characteristics from a portfolio of N debt instruments (i.e. defaultable bonds). A quite general example would be as follow:

Tranche 1: Absorbs all credit losses from the portfolio during the life of the CDO until they have reached $p_1\%$ of the total bond principal.

Tranche 2: It has $p_2\%$ of the principal and absorbs all losses in excess of $p_1\%$ of the principal up to a maximum of $q_2\% = (p_1 + p_2)\%$ of the principal.

Generally, Tranche i is responsible for defaults between $q_i\%$ and $q_{i+1}\%$. (same as losses cause of the same principal L).

The M tranche has $1 - \sum_{i=1}^{M-1} p_i\%$ of the principal and absorbs all losses in excess of $q_M\% = \sum_{i=1}^{M-1} p_i\%$ of the principal. Notice $\sum_{i=1}^M p_i\% = 1$. ($q_i + p_i = q_{i+1}$).

Each Tranche has specific yields (r_i), which represent the rates of interest paid to tranche holders. These rates are paid on the balance of the principal remaining in the tranche after losses have been paid. The principal to which the promised payments are applied declines as defaults occur. Tranche 1 is sometimes referred to as toxic waste while Tranche N by contrast is usually given an Aaa rating.

Notice that the present value of the expected cost of defaults for this tranche is the sum of the cost of defaults for n^{th} to default CDS for values of n between $q_i\%$ and $q_{i+1}\%$. Suppose that there is a promised percentage payment of r_i at time τ . In our case the payment is:

1. $p_i\% \cdot L \cdot r_i$ with probability $1 - P(q_i, \tau)$.

2. $(p_i\% - 1\%) \cdot L \cdot r_i$ with probability $\pi_\tau(q_i)$.
3. $(p_i\% - 2\%) \cdot L \cdot r_i$ with probability $\pi_\tau(q_i + 1)$.
4. So on.

Proposition 2. *The expected payment, payoff, for tranche i , would be:*

$$p_i \cdot r_i \cdot L \cdot [1 - P(q_i, \tau)] + \sum_{j=1}^{p_i-1} [(p_i - j) \cdot r_i \cdot L \cdot \pi_\tau(q_i + j - 1)]. \quad (17)$$

The present value (t) of tranche i can be computed, based on a Black-Scholes argument as:

$$\begin{aligned} & \int_t^T e^{-r \cdot (s-t)} \cdot (p_i \cdot r_i \cdot L) ds - \int_t^T e^{-r \cdot (s-t)} \cdot (p_i \cdot r_i \cdot L) \cdot f_{q_i}(s) ds \\ & + \sum_{j=1}^{p_i-1} \left\{ \int_t^T e^{-r \cdot (s-t)} \cdot [(p_i - j) \cdot r_i \cdot L] \cdot f_{q_i+j-1}(s) ds \right\}. \end{aligned} \quad (18)$$

Proposition 3. *The distribution of losses $Lo(t)$ for tranche i can be obtained by noticing the following relation:*

$$P(Lo(t) = L \cdot q_i + s) = \frac{\pi_t(q_i + s)}{\sum_{l=0} \pi_t(q_i + l)} \quad (19)$$

where π_t were defined in equations 3-8.

Similarly to pricing CDSs, the price of tranches are explicit functions of joint probabilities of defaults, therefore the densities $f_j(t)$ are the key element in those equation. This density can be computed from the solution to the Fokker-Planck equation (see Theorem 1, equation 8 and remark 1).

3 Conclusion

A partial differential equations with absorbing boundary conditions is shown to provide the joint distribution of minimums and endpoints of a vector of correlated diffusion processes. The implications of this PDE for pricing derivatives is presented along the lines of Merton 1974 and Black-Scholes 1976 structural models. The closed form solution for this joint distribution in the cases of one and two dimensional cases is described..

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