Portfolio optimization in a multidimensional structural-default model with a focus on private equity

Marcos Escobar
Department of Mathematics,
Ryerson University, Toronto,
245 Church Street, Toronto, ON, Canada,
email: escobar@ryerson.ca,

Peter Hieber
Lehrstuhl für Finanzmathematik,
Technische Universität München,
Parkring 11, 85748 Garching-Hochbrück, Germany,
email: hieber@tum.de,

Matthias Scherer
Lehrstuhl für Finanzmathematik,
Technische Universität München,
Parkring 11, 85748 Garching-Hochbrück, Germany,
email: scherer@tum.de,

Luis Seco
Department of Mathematics,
University of Toronto,
40 St. George Street, Toronto, ON, Canada,
email: seco@math.utoronto.ca.

Abstract

Investments in various asset classes, such as e.g. private equity or hedge funds, are prone to default risk. This risk should be taken into account for the calculation of individual investment opportunities and for optimal portfolio selection. The correspondent literature on portfolio optimization, however, mostly disregards default risk and accordingly skewed return distributions. This paper presents a realistic and tractable framework for a portfolio optimization including default risk. A specific focus lies on private equity investments. Default events are modeled by means of a Merton- or Black-Cox-type structural model. On a portfolio level, the mean and covariance of the resulting return distribution can be derived analytically, allowing for a classical mean-variance optimization. To include tail risk, we additionally present a Monte-Carlo simulation for a mean-$CVaR$ optimization. The paper concludes with an application to unlisted private equity and compares the results with a model proposed by [Hamada [1972]] that does not explicitly consider default risk.

Keywords

Portfolio optimization; structural-default model; private equity.

Marcos Escobar is a professor at the Department of Mathematics at Ryerson University. Since 2004, he holds a Ph.D. from the Department of Mathematics at the University of Toronto. His research focus lies on Stochastic Processes, Statistics, and Finance.

Peter Hieber is a Ph.D. candidate at TU München. After completing a B.Sc. in the field of (Financial-) Mathematics with minor Economics and a M.Sc. in Finance and Information Management, his research interests now lie on option pricing, dependence modelling, and credit-risk management.

Matthias Scherer is a professor for Mathematical Finance at TU München. He coordinates the elite-graduate programme Finance and Information Management and teaches various courses in mathematical finance. His research focus lies on credit-risk modelling, multivariate models, and dependence concepts.

Luis Seco received his Ph.D. from Princeton University in 1989. In 1996, he founded Risklab Toronto as a joint initiative between the University of Toronto and Algorithmic Inc. He is currently a professor at the Department of Mathematics at the University of Toronto. He is president and CEO of Sigma Analysis, a portfolio management firm which serves the institutional alternative investment sector worldwide.

The benefits and risks of investing in private equity are extensively (and controversely) discussed in the literature. While, e.g., [Ljungqvist, Richardson [2003]] find positive excess returns of private equity on stocks, other authors attribute those excess returns to wrongly estimated risk, e.g., [Phalippou, Gottschalg [2009]], or to a bias in the available data, e.g., [Cochrane [2005]]. Vast agreement exists on the fact that individual private equity investments are quite risky and a vital part for successfully investing in private equity is diversification. However, portfolio optimization in this context has turned out to be difficult. In contrast to standard bond or stock portfolios, one has to face at least three additional difficulties, as pointed out in, e.g., [Moskowitz [2002]].

First of all, large individual investments often contribute a significant fraction to a private equity portfolio. The result is idiosyncratic risk that cannot be fully eliminated through diversification. The second difficulty is the, contrasting standard stocks, high probability of failure for private equity projects. [Gurung, Lerner [2008]] find that about 7% of all private-equity deals undergo financial distress. This number is about twice as high as for US publicly traded firms. Finally, there is a lack of reliable data on private equity. Especially for unlisted private equity (which according to Thomson's Security Data Corporation (SDC) accounts for about two thirds of all reported aquisitions), one has to rely on only few information. Reported Net Asset Values only occasionally reflect the true values, see, e.g., [Buchner et al. [2008]]. Several authors worked on the last issue, tackling the problem of unlisted private equity. [Takahashi, Alexander [2002]] and [Malherbe [2004]] rely on indirectly observable data; [Metrick, Yasuda [2010]] use (as far as possible) publicly available cashflow and accounting information, while [Braun et al. [2011]] rely on proprietary databases of two international private equity funds-of-funds.

In this paper, we address the first two problems. We propose a model that includes default risk and suggest an optimization procedure that can be applied to minimize idiosyncratic risk. In our model, equity is interpreted as a call option on the firm's assets, its liabilities being the corresponding strike price. Default in this setting can take place at maturity, see, e.g., [Merton [1974]], or continuously, see, e.g., [Black, Cox [1976]] and the many generalizations of these seminal papers. Hence, in contrast to existing models, mostly relying on the normality assumption of the CAPM (see, e.g., [Hamada [1972]] and further extensions), we take into account the default probability of private equity investments.

Concerning optimization procedures for minimizing idiosyncratic risk of private equity, only few literature is available. Closest to our paper are [Agarwal, Naik [2003]] and [Amenc, Martellini [2002]] on portfolio optimization of hedge funds. However, their approach relies on available equity time series and does not include default risk. In our framework, we derive mean and covariance of the resulting returns analytically and carry out a mean-variance optimization. Yet, relying only on the first two (mixed) moments has been criticized in the literature on portfolio optimization, see, e.g., [Fung, Hsieh [2000]], [Mukherji [2006]]. Especially for risk assessment and management, the probability of large losses should be included in the optimization procedure. Motivated by these concerns, we show in a second step how the Conditional Value-at-Risk ($CVaR$) framework can be applied to construct private equity portfolios. This framework explicitly accounts for tail risk.

The paper is organized as follows. In the first two sections we introduce the model and derive mean and covariance of the return distribution in closed form. After that, a Monte-Carlo algorithm, originally proposed by [Hull, White [2005]] for the pricing of portfolio credit derivatives, is illustrated and adapted to the present situation. This algorithm is used for computing the portfolio $CVaR$. The optimization algorithms are presented consequently. Finally, these algorithms are applied in a case study on private equity. The last section concludes and discusses possible extensions.

Model description and mathematical results

We consider n companies and define $A_i(t)$, $i \in \{1, \ldots, n\}$, as the asset-value of company i at time $t \in [0, T_i]$. Following [Black, Scholes [1973]], the asset-value process is modelled by the Geometric Brownian motion

$$dA_i(t) = A_i(t)(\mu_i dt + \sigma_i dW_i(t)), \; A_i(0) = A_{i,0}, \tag{1}$$

where $W_i(t)$ is a standard Brownian motion and $corr(W_i(t), W_j(t)) := \rho_{ij}$. The riskless interest rate is denoted by r. To simplify the calibration of the model one often relies on a so called factor-model construction. In this case, the (normally distributed) returns $\ln A_i(t)$ are interpreted as a weighted sum of certain risk factors, i.e.

$$\ln(A_i(t)) = \alpha_i(t) + \sum_{l=1}^{L} \beta_{i,l} M_l(t) + \gamma_i Z_i(t), \tag{2}$$

where $M_l(t)$, $l \in \{1, \ldots, L\}$, are market or industry wide factors modeling certain cyclical or industry specific events and $Z_i(t)$ is the idiosyncratic risk of firm i. These processes are independent Brownian motions and $\sum_{l=1}^{L} \beta_{i,l}^2 < \sigma_i^2$. To match Equations (1) and (2), we set $\alpha_i(t) := \ln(A_{i,0}) + \left(\mu_i - \frac{1}{2}\sigma_i^2\right) t$ and $\gamma_i := (\sigma_i^2 - \sum_{l=1}^{L} \beta_{i,l}^2)^{\frac{1}{2}}$. The correlation between the Brownian motions in Equation (1) is now given by $corr(W_i(t), W_j(t)) =: \rho_{ij} = \sum_{l=1}^{L} \beta_{i,l}^2 / \sigma_i^2$.

Black-Cox type default model

The seminal assumption in the structural-default model of [Black, Cox [1976]] is that default might be triggered continuously. More precisely, company i defaults whenever the value of its assets $A_i(t)$ falls below the face value of debt D_i, which in our framework is assumed to be constant over time. This option-like valuation for highly leveraged firms is empirically supported (see, e.g., [Green [1984]], [Arzac [1996]]). [Schaefer, Strebulaev [2008]] show that structural models provide quite accurate predictions of the sensitivity of returns to changes in the value of equity. Consequently, the default time of firm i is defined as $\tau_i := \inf\{t \geq 0 : A_i(t) \leq D_i\}$. Note that, if $\tau_i < T_i$, company i defaults before maturity T_i.

In a second step, we consider an investor who composes a portfolio of the given companies/investment opportunities. This investor favors a specific investment horizon $\mathscr{T} < \min\{T_1, \ldots, T_n\}$. The value of her position in company i can be calculated through a *knock-out-barrier-option* DOC with threshold level D_i. The maturity T_i of this option is given by the duration of the respective investment. The corresponding valuation formula of the DOC, see [Reiner, Rubinstein [1991]], is for the reader's convenience recalled in the Appendix. The value of company i during the investment period $[0, \mathscr{T}]$ is then given by

$$\Pi_i(t) := 1_{\{\tau_i > t\}} DOC\Big(A_i(t), T_i - t, \theta_i\Big), \tag{3}$$

where $\theta_i := (\sigma_i, D_i, r)$. Assuming the fractional weights $0 \leq x_i \leq 1$, where $x_1 + \ldots + x_n = 1$, invested in company i, the portfolio value at time t is $\Pi(t) := \sum_{i=1}^n x_i \; \Pi_i(t)$. The composition of this portfolio is assumed to be static on $[0, \mathscr{T}]$.

Merton type default model

For a later comparison to the Black-Cox model, this section applies the same setting to the Merton model, see [Merton [1974]]. In this framework, default is possible at maturity T_i only. At time 0, this leads to the payoff of a European call option with strike D_i. This expression is known in closed form, see, e.g., [Black, Scholes [1973]]:

$$\begin{aligned} C(A_i(t), T_i - t) &= e^{-r(T_i - t)} \mathbb{E}_{\mathbb{Q}}\Big[\max\big(A_i(T_i) - D_i, 0\big)\Big] \\ &= A_i(t)\; \Phi\big(d_1\big(A_i(t), T_i - t\big)\big) - D_i e^{-r(T_i - t)} \Phi\big(d_2\big(A_i(t), T_i - t\big)\big), \end{aligned} \tag{4}$$

$$\begin{aligned} d_1\big(A_i(t), T_i - t\big) &= \frac{\ln(A_i(t)/D_i) + (r + \frac{1}{2}\sigma_i^2)(T_i - t)}{\sigma_i \sqrt{T_i - t}}, \\ d_2\big(A_i(t), T_i - t\big) &= d_1\big(A_i(t), T_i - t\big) - \sigma_i \sqrt{T_i - t}, \end{aligned}$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of a standard normal distribution. Similar to Equation (3), we again get the value for company i during the investment period $t \in [0, \mathscr{T}]$ as:

$$\Pi_i(t) := C\Big(A_i(t), T_i - t\Big). \tag{5}$$

Mathematical results

Mean and covariance of the return vector

The subsequent mean-variance optimization requires the mean of the returns and the covariance between the returns of the portfolio described in the last section. We define discrete returns over the investment period $[0, \mathscr{T}]$, denoted $R_i := \Pi_i(\mathscr{T})/\Pi_i(0) - 1$. The return's variance is given by $\mathrm{Var}(R_i) = \mathbb{E}[R_i^2] - \mathbb{E}[R_i]^2$. To derive the covariance matrix, we furthermore need an extension that generalizes the notion of default correlation of [Lucas [1995]] and [Zhou [2001]]. We consider

$$\rho_{ij}^R := \mathrm{corr}(R_i, R_j) = \frac{\mathbb{E}[R_i R_j] - \mathbb{E}[R_i]\ \mathbb{E}[R_j]}{\sqrt{\mathrm{Var}(R_i)\mathrm{Var}(R_j)}}. \tag{6}$$

Thus, we need in both the Merton and the Black-Cox models the expectations $\mathbb{E}[R_i]$, $\mathbb{E}[R_i^2]$, and $\mathbb{E}[R_i R_j]$. In the Black-Cox model, $\mathbb{E}[R_i]$ and $\mathbb{E}[R_i^2]$ can be calulated using a result of [Rubinstein [1992]], see Theorem A.2 in the Appendix. The expectation $\mathbb{E}[R_i R_j]$ was first derived by [He et al. [1998]] for the simpler case $R_i = 1_{\{\tau_i > t\}}$. Following his ideas we obtain the result for the return R_i defined above. These findings are presented in detail in the Appendix, Theorem A.2. In a Merton-type model, the same derivations are possible. All results are presented in the Appendix, Theorem A.3.

For highly-leveraged investments, such as e.g. private-equity deals, both the Merton and Black-Cox model lead to a non-normal return distribution. The more conservative assumption, with a higher probability of default, is the Black-Cox model. Given the asset-value process, this section showed how to derive mean $\mu^{R_i} := \mathbb{E}[R_i]$ and covariance $\Sigma^R := Cov(R_i, R_j) = \rho_{ij}^R \sqrt{\mathrm{Var}(R_i)\mathrm{Var}(R_j)}$ of equity returns. Those results can later be used for a mean-variance optimization. As discussed in, e.g. [Mukherji [2006]], the portfolio (co)variance is often not an adequate measure of portfolio risk. However, the calculation of risk measures like $CVaR$ has to be done numerically. Thus, the following section introduces a Monte-Carlo approach to calculate risk measures that account for the increased tail risk.

Monte-Carlo Simulation to estimate $CVaR$

The shortcoming of the approach in the previous section is the fact that a covariance matrix does not capture the whole distribution and dependence structure of the presented model. Multiple defaults result in heavy tails of the portfolio return distribution. This effect can be captured by a mean-$CVaR$ optimization. Hull and White (2001, 2005) proposed a Monte-Carlo algorithm for the pricing of CDOs in a multidimensional Black-Cox model. A slightly modified version of this algorithm is described below.

Hull and White assume that asset-values can only be observed at k discrete points in time, for example at invervals agreeing with quarterly accounting reports (or scheduled premium payment dates). Default is only possible at these points in time. Their idea is to subsequently adjust the threshold level at each of the k discrete points to account for the (continuous) Black-Cox definition of default, see [Hull, White [2001]] for details. The result is a slightly raised barrier $D_i^* = [D_{i1}^*, D_{i2}^*, \dots, D_{ik}^*]$ and a discrete model that inherits the marginal probabilies of default up to the respective discrete dates from the Black-Cox model.

Including dependence by a one factor Gaussian model (Equation (2) with $L = 1$), we get the following Monte-Carlo simulation. The indices h, respectively i and j, represent the simulation run index, respectively the company and discretization index.

1. *Draw a set of independent standard normally distributed random variables* $\Delta M^{j,h}$ *and* $\Delta Z^{i,j,h}$ *for* $h \in \{1, 2, \dots, N\}$*,* $i \in \{1, 2, \dots, n\}$*, and* $j \in \{1, 2, \dots, k\}$*.*
2. *Calculate the adjusted barriers* $D_i^* = [D_{i1}^*, D_{i2}^*, \dots, D_{ik}^*]$*.*
3. *Simulate* N *asset paths by* $M^0 = 0$*,* $Z^{0,j} = 0$*, and*
   - $M^j = M^{j-1} + \Delta M^j \sqrt{\frac{\mathscr{T}}{k}}$,
   - $Z^{i,j} = Z^{i-1,j} + \Delta Z^{i,j} \sqrt{\frac{\mathscr{T}}{k}}$,
   - $\ln(A_i(\mathscr{T}j/k)) = \alpha_i(\mathscr{T}j/k) + \beta_i M^j + \gamma_i Z^{i,j}$.
4. *Get the realizations of the portfolio values depending on the weights* x_i *(*$\theta_i = (\sigma_i, D_i^*, r)$*)*

$$\Pi(\mathscr{T}) \approx \sum_{i=1}^{n} x_i \; 1_{\left\{ \min_{\{1 \le j \le k\}} A_i\left(\frac{j}{k}\mathscr{T}\right) > D_{ij}^* \right\}} DOC\Big(A_i(\mathscr{T}), T_i - \mathscr{T}, \theta_i\Big).$$

5. *Get the portfolio value at time 0 and obtain* N *discrete portfolio returns depending on the weights* x_i

$$\Pi(0) = \sum_{i=1}^{n} x_i \; DOC\Big(A_i(0), T_i, \theta_i\Big), \;\; R = \Pi(\mathscr{T})/\Pi(0) - 1.$$

6. *Optimize the portfolio on a given risk measure, i.e. find optimal weights* x_i*.*

In comparison to a naïve Monte-Carlo simulation (no adjustment of the barrier), this algorithm has the advantage that even for small values of k (e.g., $k = 10$) not only the default probabilities, but also the linear correlation, variance, and expectation of the continuous model are almost exactly replicated. As the Merton model allows for default only at maturity, simulating a Merton model is much faster. Here, the barrier does not have to be adjusted and the above algorithm can easily be modified.

Portfolio optimization

Having introduced an analytical approach for mean and covariance and a Monte-Carlo simulation that provides a tool to estimate a portfolio $CVaR$, this section describes a portfolio optimization using variance and $CVaR$ as measures of risk. We assume a portfolio of n companies with threshold level D_i, an asset-value process with mean μ_i and volatility σ_i ($i \in \{1, \ldots, n\}$), and covariance matrix Σ. As described in the first section, the companies are monitored over the periods $[0, T_i]$. The investment horizon is $[0, \mathscr{T}]$ with $\mathscr{T} < \min\{T_1, \ldots, T_n\}$.

The portfolio optimization problem, including an aspired risk aversion parameter λ, is given by:

$$(P) = \begin{cases} \max x^T \mu^{R_i} - \frac{\lambda}{2}\, risk \\ s.t.\ x_i \geq 0,\ \sum_{i=1}^{n} x_i = 1 \end{cases}$$

with $risk$ specified as $x^T \Sigma^R x$ for the mean-variance and $CVaR$ for the mean-$CVaR$-optimization respectively. For a definition of $CVaR$ see, e.g., [Krokhmal et al. [2002]]. The advantage of both algorithms is their numerical stability. The variance optimization is a quadratic optimization problem; the mean-$CVaR$ optimization can be transformed in a linear problem following [Krokhmal et al. [2002]]. Note that closed-form expressions for the portfolio weights are available in a mean-variance optimization when the constraint of non-negative weights is disregarded, see, e.g., [Ferguson, Leistikon, Yu [2009]].

The calculation of the parameters $\mu_i^{R_i}$ and Σ^R was discussed before; $CVaR$ can be estimated by the presented Monte-Carlo algorithm. These risk measures can now be used as input parameters to finally get the optimal weights x_i of the companies in the portfolio. This procedure is demonstrated in the following case study on private equity.

Case Study: Private equity portfolio optimization

In this section, both models are applied using reasonable input parameters. To study the effect of our extensions, we compare those models to a model with normality assumption and leverage, but disregarding default risk. To do so, consider Hamada's equation, see [Hamada [1972]]. This equation is based on the following considerations. Assume a company (w.l.o.g. debt + equity = 1) with asset-value process given by Equation (1). This company can either be unleveraged, in which case the equity holders receive an expected return of $A_i(\mathscr{T})/A_{i,0}$ during the investment period $[0, \mathscr{T}]$. Otherwise, it can be leveraged with debt ratio D. In this case, the equity is $1 - D$ and the equity investor has to pay $rD\mathscr{T}$ to the bondholders. Excluding default risk, the equity return is $R_i = (A_i(\mathscr{T})/A_{i,0} - rD\mathscr{T})/(1 - D)$. Using the same notation as in the first section, expecteted return and covariance are given by $\mu^{R_i} := \mathbb{E}[R_i] = (\mu - rD)\mathscr{T}/(1 - D)$ and $\Sigma^R = \mathscr{T}^2/(1 - D)^2\ \sigma_i \sigma_j \rho_{ij}$.

For the case study, we choose a portfolio of three different companies with the same asset-value process, but different levels of leverage. We take as base case[1] the parameters of [Gwangheon, Sarkar [2007]] and use $\sigma_i = 20\%$, $r = 3\%$, and $\rho_{ij} = 0.6$, for $i, j \in \{1, 2, 3\}$. We take $\mathscr{T} = 1$ year as investment horizon; the duration of the private equity investment is $T = 10$ years, following [Malherbe [2004]]. We consider the companies „Safe“ ($\mu_1 = 4\%$, $D_1 = 0.20$), „Normal“ ($\mu_2 = 9\%$, $D_2 = 0.70$), and „Distressed“ ($\mu_3 = 10\%$, $D_3 = 0.85$). We then compute mean and covariance as described before and carry out the portfolio optimization for the three models Black-Cox, Merton, and Hamada. Mean and variance for „Safe“ are equal for all three models; however its correlation to „Distressed“ is slightly different: 0.55 (Black-Cox model), 0.59 (Merton model), and 0.60 (Hamada model).

Figure 1 shows the results of the mean-variance optimization. The black line indicates $\lambda = 1$. „Distressed“ is the riskiest company with the highest leverage; „Safe“ is the least risky of the three. The investment decisions of the Hamada (Figure 1, below) and Black-Cox model (Figure 1, above) are quite similar. However, lower correlation in the Black-Cox model disperses more on the different companies. The investment in „Distressed“ is slightly lower in a Black-Cox model: For $\lambda = 1$ its share is 11.9% (Black-Cox) compared to 13.7% (Hamada). Obviously the Merton model (Figure 1, middle) leads to the riskiest investment decision. The fact that default is only possible at maturity T_i (therefore no default in $[0, \mathscr{T}]$) combined with a lower correlation than in the Hamada model seems to favour a higher share in the high return/high risk companies.

The mean-$CVaR$ optimization based on a 90% $CVaR$ is presented in Figure 2. Again the case $\lambda = 1$ is highlighted. The Black-Cox model now proposes a much more conservative investment than the other two. For risk aversion parameters greater than 1, the share of „Safe“ exceeds 80%. The case $\lambda = 1$ indicates that both the Merton and the Hamada model lead to riskier investment decisions. Due to the fact that default is only possible at maturity (Merton) and the normality of the return distribution (Hamada), their share of 21.2% (Merton) and 42.1% (Hamada) in „Safe“ is much lower than the 82.3% of the Black-Cox model.

The results are further fortified by the efficient frontiers. Figure 3 shows the efficient investment frontier for the mean-variance optimization (above) and the mean-$CVaR$ optimization (below). One observes that for both, mean-variance and mean-$CVaR$ optimization, the Black-Cox model gives the most conservative results. The Merton model with its assumption of a default possibility only at maturity seems to underestimate the portfolio risk. Even the Hamada model, that does not take default risk into account, gets higher numbers for the risk measures variance and $CVaR$. This is in line with [Hao [2006]] who shows empirically that barrier options fit default probabilities better than standard call options. All examples show that default risk is not negligible for portfolio optimization and can lead to far different investment decisions.

[1] For an overview on calibration procedures for structural default models, see, e.g., [Hao [2006]].

Conclusion

The contribution of our paper to the existing literature is threefold. Firstly, we propose an extension of existing leverage models (see, e.g, [Hamada [1972]] and further extensions, e.g., [Conine [1980]] and [Cohen [2007]]) by including default risk via a structural approach. Secondly, this paper is, to our knowledge, the first framework for a portfolio optimization in a multidimensional structural-default model environment. Thirdly, our model can handle the case of unlisted private equity which is rarely treated in the literature.

We presented a model that includes default by a real option approach. Equity is seen as a call option on the firm's total assets. Defaults can either occur continuously (see [Black, Cox [1976]]) or solely at maturity (see [Merton [1974]]). In this framework, we analytically derived mean and covariance of discrete returns and used these results for a mean-variance optimization. As this does not account for extreme tail risk we, in a second step, presented a concept to perform a mean-$CVaR$ optimization using an adequately modified Monte-Carlo algorithm from [Hull, White [2005]].

In a case study for unlisted private equity, we illustrate that including default risk in a portfolio optimization might lead to far different results. The Black-Cox model is (with its higher probability of default) a more conservative assumption than the Hamada model which does not include default risk. The Merton model seems to understate the portfolio risk due to its questionable assumption of no default prior to maturity.

Further applications of this portfolio optimization technique are other risky investment classes, e.g., hedge funds or real estate. This rather simple model can be extended in several ways to include more realistic assumptions. One possibility could be a random threshold level following, e.g., [Finger [2002]], [Giesecke, Goldberg [2004]], and [Fouque et al. [2008]]. Furthermore it might make sense to include taxes, transaction and/or information costs.

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A Mathematical results

The Appendix displays the components $\mathbb{E}[R_i]$, $\mathbb{E}[R_i^2]$, and $\mathbb{E}[R_iR_j]$ which are required to calculate the variance $\mathrm{Var}(R_i)$, correlation ρ_{ij}^R (definition, see above), covariance matrix $\Sigma^R := \rho_{ij}^R\sqrt{Var(R_i)Var(R_j)}$, and expectation $\mu^{R_i} := \mathbb{E}[R_i]$ of the returns $R_i = \Pi_i(\mathscr{T})/\Pi_i(0) - 1$. As a first step, observe

$$\rho_{ij}^R = \frac{\mathbb{E}[R_iR_j] - \mathbb{E}[R_i]\mathbb{E}[R_j]}{\sqrt{Var(R_i)Var(R_j)}} = \frac{\mathbb{E}[(R_i+1)(R_j+1)] - (\mathbb{E}[R_i]+1)(\mathbb{E}[R_j]+1)}{\sqrt{\mathrm{Var}(R_i)\mathrm{Var}(R_j)}}.$$

Theorems A.2 and A.3 give the expression for $\mathbb{E}[(R_i+1)(R_j+1)]$.

A.1 Black-Cox model

The price of a *knock-out-barrier-option*, see [Reiner, Rubinstein [1991]], is presented in Theorem A.1.

Theorem A.1 (Knock-out-barrier-option)
Let $A_i(t)$ be an asset-value process, modelled as a Geometric Brownian motion with volatility σ_i, with strike and knock-out barrier D_i. The time to maturity is T_i, r the risk-free interest rate. Then, the value of a knock-out-barrier-option is given by

$$\begin{aligned} DOC(A_i(0),T_i) =& A_i(0)\Big(\Phi\big(d_1(A_i(0),T_i)\big) - \left(\frac{D_i}{A_i(0)}\right)^{2r/\sigma_i^2+1}\Phi\big(d_2(A_i(0),T_i)\big)\Big) \\ &- D_i e^{-rT_i}\Big(\Phi\big(d_1(A_i(0),T_i) - \sigma_i\sqrt{T_i}\big) - \left(\frac{D_i}{A_i(0)}\right)^{2r/\sigma_i^2+1}\Phi\big(d_2(A_i(0),T_i) - \sigma_i\sqrt{T_i}\big)\Big) \end{aligned} \tag{7}$$

where $d_1(A_i(0),T_i) = (\ln(A_i(0)/D_i) + (r+\frac{1}{2}\sigma_i^2)T_i)/(\sigma_i\sqrt{T_i})$, $d_2(A_i(0),T_i) = (\ln(D_i/A_i(0)) + (r+\frac{1}{2}\sigma_i^2)T_i)/(\sigma_i\sqrt{T_i})$, and $\Phi(\cdot)$ denotes the cumulative distribution function of a standard normal distribution.

Theorem A.2 (Moment calculation in the Black-Cox model)
Let $R_i = \Pi_i(\mathscr{T})/\Pi_i(0) - 1$ *on* $(\Omega, \mathcal{F}_t, \mathcal{P})$, $\rho_{ij} = corr(A_i, A_j)$, *and* $a_i(t) := \ln(A_i(t))$, $\forall t$. *Then*

$$\mathbb{E}[R_i] = \frac{1}{\sigma_i\sqrt{\mathscr{T}}} \int_{\ln(D_i)}^{\infty} \frac{DOC(A_i(\mathscr{T}), T_i - \mathscr{T})}{DOC(A_i(0), T_i)} \ \phi\left(\frac{-a_i(\mathscr{T}) - (\mu_i - \frac{1}{2}\sigma_i^2)\mathscr{T}}{\sigma_i\sqrt{\mathscr{T}}}\right) \cdot \left(1 - e^{-\frac{4\ln(D_i)^2 - 4\ln(D_i)a_i(\mathscr{T})}{2\sigma_i^2\mathscr{T}}}\right) da_i(\mathscr{T}) - 1, \tag{8}$$

$$\mathbb{E}[R_i^2] = \frac{1}{\sigma_i\sqrt{\mathscr{T}}} \int_{\ln(D_i)}^{\infty} \left(\frac{DOC(A_i(\mathscr{T}), T_i - \mathscr{T})}{DOC(A_i(0), T_i)}\right)^2 \ \phi\left(\frac{-a_i - (\mu_i - \frac{1}{2}\sigma_i^2)\mathscr{T}}{\sigma_i\sqrt{\mathscr{T}}}\right) \cdot \left(1 - e^{-\frac{4\ln(D_i)^2 - 4\ln(D_i)a_i(\mathscr{T})}{2\sigma_i^2\mathscr{T}}}\right) da_i(\mathscr{T}) - 2\mathbb{E}[R_i] - 1, \tag{9}$$

where $\phi(\cdot)$ *denotes the density of a standard normal distribution.*

$$\mathbb{E}[(R_i + 1)(R_j + 1)] = \frac{e^{c_1\ln(D_i) + c_2\ln(D_j) + b\mathscr{T}}}{\sigma_i\sigma_j\sqrt{1 - \rho^2}} \int_{\ln(D_i)}^{\infty}\int_{\ln(D_j)}^{\infty} \left(\frac{DOC(A_i(\mathscr{T}), T_i - \mathscr{T})}{DOC(A_i(0), T_i)}\right) \cdot \left(\frac{DOC(A_j(\mathscr{T}), T_j - \mathscr{T})}{DOC(A_j(0), T_j)}\right) \cdot h\big(a_i(\mathscr{T}), a_j(\mathscr{T}), \mathscr{T}; \rho_{ij}\big) da_j(\mathscr{T})\, da_i(\mathscr{T}),$$

where

$$h(x_i, x_j, \mathscr{T}; \rho_{ij}) = \frac{2}{\beta\mathscr{T}} \sum_{n=1}^{\infty} e^{-\frac{r^2 + r_0^2}{2t}} \sin\left(\frac{n\pi\theta_0}{\beta}\right) \sin\left(\frac{n\pi\theta}{\beta}\right) I_{n\pi/\beta}\left(\frac{rr_0}{\mathscr{T}}\right),$$

$$c_1 = \frac{\left(\mu_i - \frac{1}{2}\sigma_i^2\right)\sigma_j - \rho_{ij}\left(\mu_j - \frac{1}{2}\sigma_j^2\right)\sigma_i}{(1 - \rho_{ij}^2)\sigma_i^2\sigma_j}, \ c_2 = \frac{\left(\mu_j - \frac{1}{2}\sigma_j^2\right)\sigma_i - \rho_{ij}\left(\mu_i - \frac{1}{2}\sigma_i^2\right)\sigma_j}{(1 - \rho_{ij}^2)\sigma_j^2\sigma_i},$$

$$b = -\left(\mu_i - \frac{1}{2}\sigma_i^2\right)c_1 - \left(\mu_j - \frac{1}{2}\sigma_j^2\right)c_2 + \frac{1}{2}\sigma_i^2c_1^2 + \rho_{ij}\sigma_i\sigma_jc_1c_2 + \frac{1}{2}\sigma_2^2ca_2^2,$$

$$\tan\beta = -\frac{1 - \rho^2}{\rho_{ij}}, \ \beta \in [0, \pi],$$

$$z_1 = \frac{1}{\sqrt{1 - \rho_{ij}^2}}\left(\left(\frac{x_i - \ln(D_i)}{\sigma_i}\right) - \rho_{ij}\left(\frac{x_j - \ln(D_j)}{\sigma_j}\right)\right), \ z_2 = \left(\frac{x_j - \ln(D_j)}{\sigma_j}\right),$$

$$z_{10} = \frac{1}{1 - \rho^2}\left(-\frac{\ln(D_i)}{\sigma_i} + \frac{\rho_{ij}\ln(D_j)}{\sigma_j}\right), \ z_{20} = -\frac{\ln(D_j)}{\sigma_j}, \ r = \sqrt{z_1^2 + z_2^2},$$

$$\tan\theta = \frac{z_1}{z_2}, \ \theta \in [0, \beta], \ r_0 = \sqrt{z_{10}^2 + z_{20}^2}, \ \tan\theta_0 = \sqrt{z_{20}^2 + z_{10}^2}, \ \theta_0 \in [0, \beta],$$

$I_\nu(z)$ *denotes the modified Bessel function of first kind.*

Proof: For the densities, see [He et al. [1998]], Formula (2.1) and Formula (2.7).

A.2 Merton model

Theorem A.3 (Moment calculation in Merton's model)
Let $R_i = \Pi_i(\mathscr{T})/\Pi_i(0) - 1$ *on* $(\Omega, \mathcal{F}_t, \mathcal{P})$, $\rho_{ij} = corr(A_i, A_j)$, *and* $a_i(t) := A_i(t)$, $\forall t$. *Then*

$$\mathbb{E}[R_i] = \frac{1}{\sigma_i\sqrt{\mathscr{T}}} \int_{-\infty}^{\infty} \frac{C(A_i(\mathscr{T}), T_i - \mathscr{T})}{C(A_i(0), T_i)} \ \phi\left(\frac{a_i(\mathscr{T}) - (\mu_i - \frac{1}{2}\sigma_i^2)\mathscr{T}}{\sigma_i\sqrt{\mathscr{T}}}\right) da_i(\mathscr{T}) - 1,$$

$$\mathbb{E}[R_i^2] = \frac{1}{\sigma_i\sqrt{\mathscr{T}}} \int_{-\infty}^{\infty} \left(\frac{C(A_i(\mathscr{T}), T_i - \mathscr{T})}{C(A_i(0), T_i)}\right)^2 \ \phi\left(\frac{a_i(\mathscr{T}) - (\mu_i - \frac{1}{2}\sigma_i^2)\mathscr{T}}{\sigma_i\sqrt{\mathscr{T}}}\right) da_i(\mathscr{T}) - 2\cdot\mathbb{E}[R_i] - 1,$$

$$\mathbb{E}[(R_i+1)(R_j+1)] = \frac{1}{2\pi\sigma_i\sigma_j\sqrt{1-\rho_{ij}^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{C(A_i(\mathscr{T}), T_i - \mathscr{T})}{C(A_i(0), T_i)} \cdot \frac{C(A_j(\mathscr{T}), T_j - \mathscr{T})}{C(A_j(0), T_j)} e^{-\frac{K(a_i(\mathscr{T}), a_j(\mathscr{T}))}{2(1-\rho_{ij}^2)}} da_j(\mathscr{T}) da_i(\mathscr{T}),$$

$$K(a_i(\mathscr{T}), a_j(\mathscr{T})) = \frac{(a_i(\mathscr{T}) - (\mu_i - \frac{1}{2}\sigma_i^2)\mathscr{T})^2}{\sigma_i^2\mathscr{T}} - \frac{2\rho_{ij}(a_i(\mathscr{T}) - (\mu_i - \frac{1}{2}\sigma_i^2)\mathscr{T})(a_j(\mathscr{T}) - (\mu_j - \frac{1}{2}\sigma_j^2)\mathscr{T})}{\sigma_i\sigma_j\mathscr{T}} + \frac{(a_j(\mathscr{T}) - (\mu_j - \frac{1}{2}\sigma_j^2)\mathscr{T})^2}{\sigma_j^2\mathscr{T}}.$$

Proof: For the first density, $a_i(\mathscr{T})$ is $\mathcal{N}((\mu_i - \frac{1}{2}\sigma_i^2)\mathscr{T}, \sigma_i\sqrt{\mathscr{T}})$ distributed; the second one is the density of correlated Brownian motions.

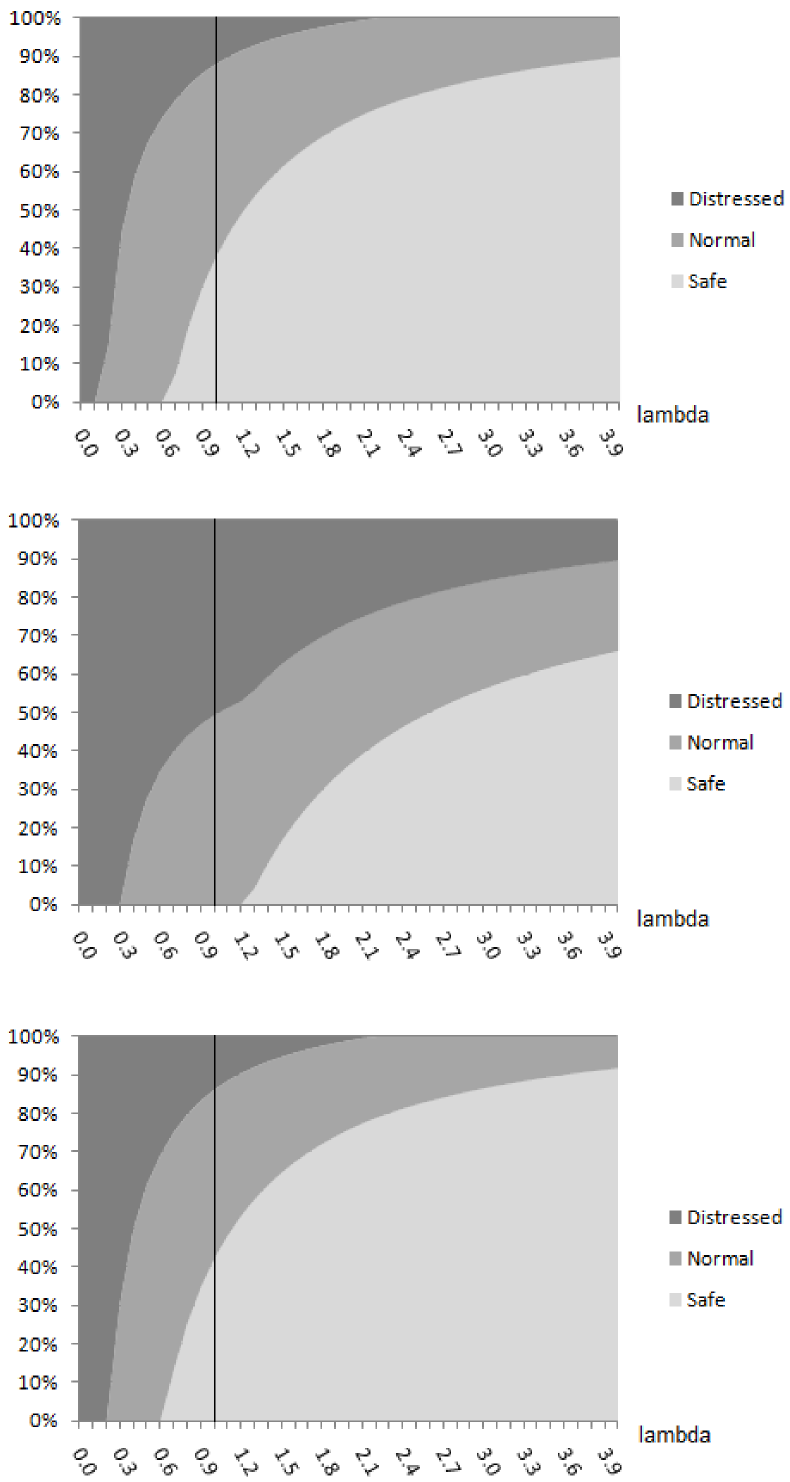

Exhibit 1 Weights from a mean-variance optimization for different levels of risk aversion λ in the Black-Cox model (above), the Merton model (middle), and the Hamada model (below).

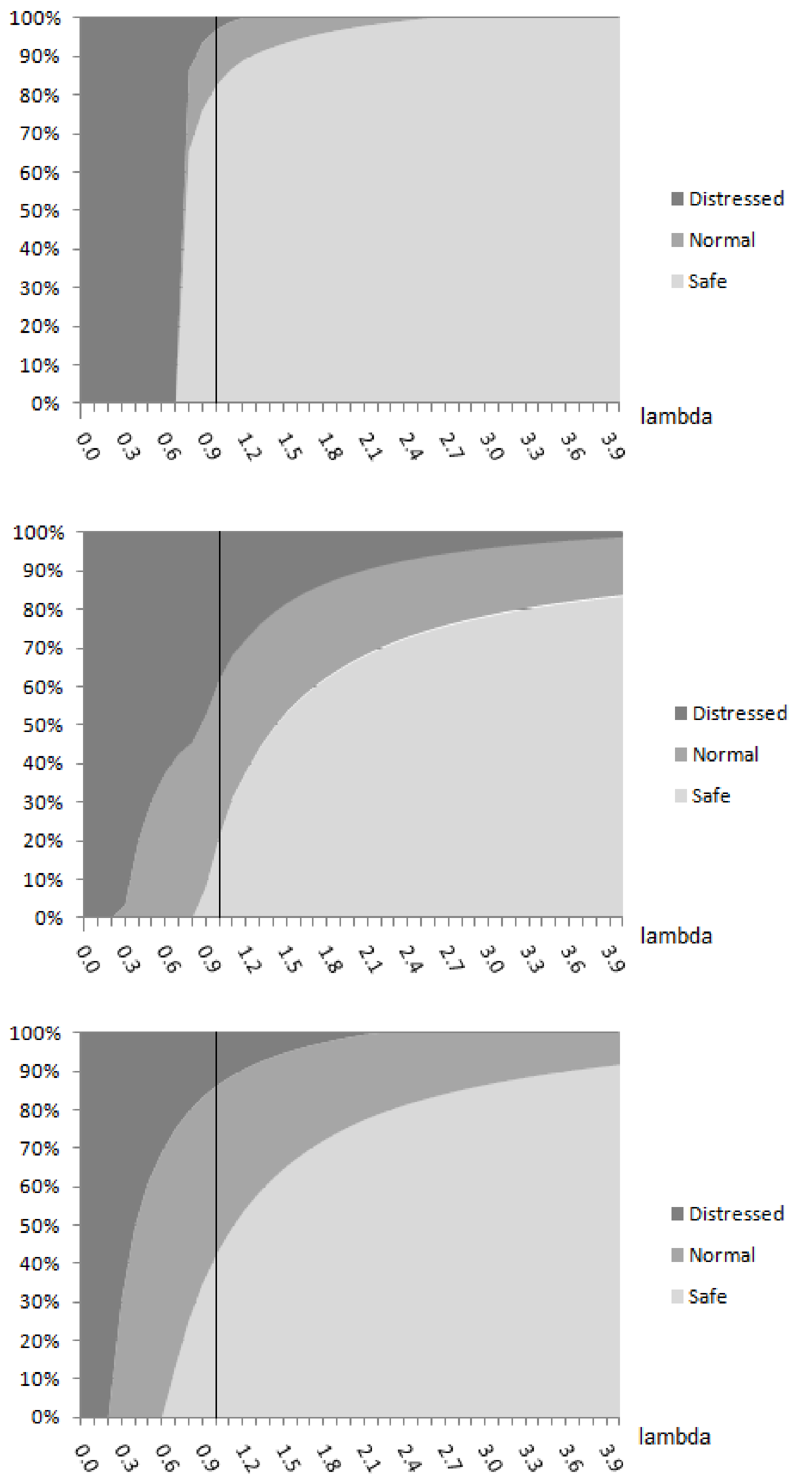

Exhibit 2 Weights from a mean-$CVaR$ optimization (90% level) for different levels of risk aversion λ in the Black-Cox model (above), the Merton model (middle), and the Hamada model (below).

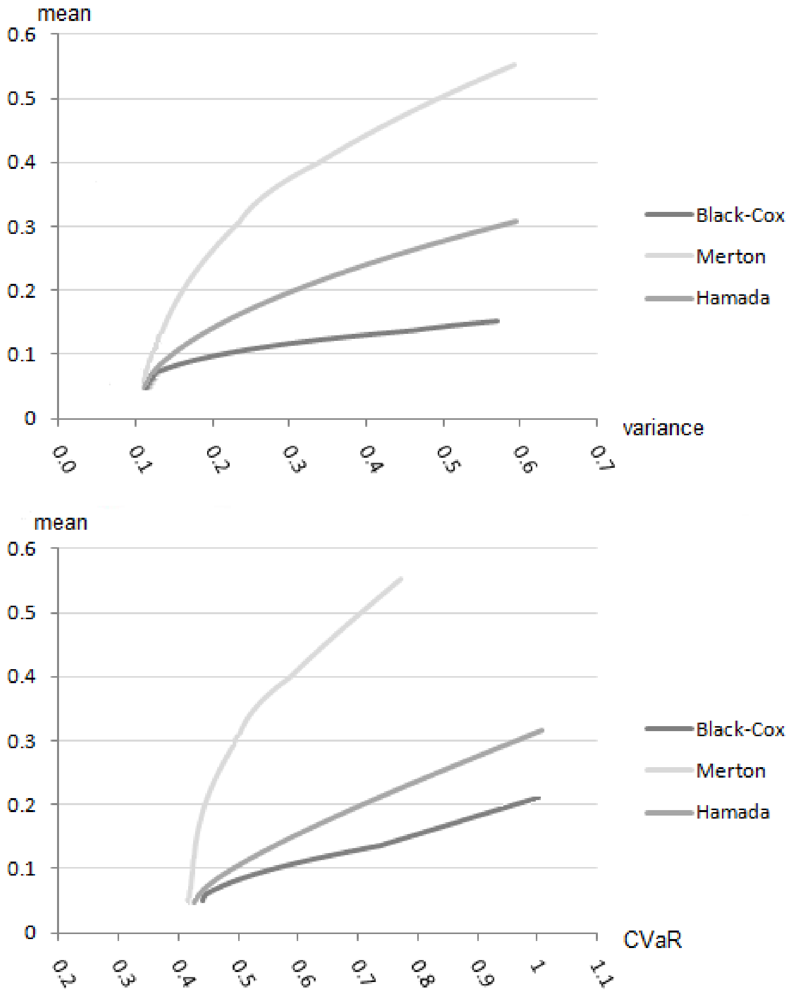

Exhibit 3 Efficient frontiers for the mean-variance optimization (above) and the mean-$CVaR$ optimization (below). One observes that in both cases, the Black-Cox model gives the most conservative results. The Merton model seems to underestimate the portfolio risk. Those risk numbers are even higher than in the Hamada model that neglects default risk.