Pricing of a CDO on stochastically correlated underlyings

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Abstract

In this paper, we propose a method to price Collateralized debt obligations (CDO) within Merton's structural model on underlyings with a stochastic mean-reverting covariance dependence. There are two key elements in our development, first we reduce dimensionality and complexity using principal component analysis on the assets' covariance matrix. Second, we approximate this continuous multidimensional structure using a tree method. Trinomial-tree models can be developed for both the principal components and the eigenvalues assuming the eigenvectors constant over time and the eigenvalues stochastic. Our method allows us to compute the joint default probabilities for k defaults of stochastically correlated underlyings and the value of CDOs in a fast manner, without having lost much accuracy. Furthermore we provide a method based on moments to estimate the parameters of the model.

Key words: stochastic covariance matrix, CDO, trinomial-trees, principal component analysis, method of moments

JEL classification: G13, C63

1 Introduction

The British Banking Association (2006) estimates in its Credit Derivatives Report 2006 that by the end of 2006 the size of the global credit derivatives market will be $ 20 trillion. Two of the most popular examples of credit derivatives are credit default swaps (CDS) and collateralised debt obligations (CDO). These credit derivative products strongly depend on the joint behavior of the underlying companies, i.e. the covariances of the assets values. There is plenty of evidence that covariances change stochastically. The recent popularity of discrete-time stochastic correlation models like the DCC (Dynamic Conditional Correlation) proposed by Engle (2002) supports this idea. On the other hand, some of the most popular procedures for pricing credit derivatives in continuous-time are non-stochastic in their correlation structures, e.g. the Linear Factor Model (LFM) (see Hull and White (2004)), the Factor-Copula Model (FCM) (see Laurent and Gregory (2003)) or the Intensity-Factor Models (see Duffie and Garleanu (2001), Hurd and

Kuznetsov (2005)). There is, nevertheless, one recent attempt, the Structural Model (see Hull et. al. (2005)), where this feature is added within a factor model framework, the main drawback being the substantial increase of complexity if stochastic volatility is added.
A rich marginal as well as dependence structure and the simplicity of the pricing scheme are the key elements for a good pricing performance. In this paper, we suggest an alternative dimension reduction method to the well studied factor-models, a stochastic principal components method. This allows us to value CDOs on assets with a stochastic mean-reverting covariance matrix which not only capture stochastic volatility of the underlyings but also stochastic correlation among them. This method is used together with Merton (1973) approach for pricing credit derivatives by working on the assets' instantaneous covariance matrix. In general there are many non-stochastic dimension reduction techniques which will require some future work, firstly extending them to stochastic process and secondly using them to create computationally-feasible methods for financial problems.
This idea of low-dimensionality is confirmed in the empirical section where the percentage of variance explained by the principal components is provided for two representative cases: the modelling of the S&P100 companies and an example on 10 chosen companies. For the later, we use the first two eigenvalues and eigenvectors from a principal components analysis of the Brownian motions of the assets. Other publications applying principal component analysis are in support of the fact that few eigenvalues (two to six) are sufficient to describe most of the variation of a pool of securities (see Alexander (2000), (2001)).
In our methodology we suggest a system of stochastic processes for the eigenvalues (based on the popular Heston (1993) volatility model) as well as for principal components, then a trinomial-tree method is developed to approximate this system. In this model the respective trees for the principal components and their volatilities, the eigenvalues, can be combined easily due to assumptions of independence among all driving brownians, those directly related to the assets as well as those driving the eigenvalues. Thus, the joint default probabilities can be found by simply multiplying the marginal probabilities. We provide a path-independent method to approximate the means of the underlying assets, which allows us to compute company values based on the values of the principal components and their respective probability. Using these asset values at maturity, the scenarios in which companies default can be calculated. As their probability is known the values of the tranches

of the CDO can be computed. Furthermore, we provide a method to fit our model to market data.

In Section 2 the model is outlined in the following sequence: relation between prices and principal components, the actual modelling of the components and then the tree approximation. We derive a trinomial-tree model for the principal components in Section 3.1 and for the eigenvalues in Section 3.2. The combined tree system is described in Section 3.3. Section 4 introduces the parameter estimation method. In the empirical Section 5, the benefits of PCA are shown in the modelling/calibration of large dimensional series, in a second part, as a representative example, a CDO is priced. We conclude in Section 6.

2 Outline of the Model

We consider m different companies and define $V_i(t)$ to be the value of the assets of a company $i,\ i \in \{1, \ldots, m\}$ at time t. The system of processes is defined on a filtered probability space $(\Omega, \mathcal{F}, \tilde{\mathcal{Q}}, \mathbb{F})$ where $\mathcal{F}_0$ contains all subsets of the $(\tilde{\mathcal{Q}}-)$ null sets of $\mathcal{F}$ and $\mathbb{F}$ is right-continuous. As we assume the market to be arbitrage-free the processes are defined under the risk neutral measure $\tilde{\mathcal{Q}}$. The dynamics of V_i are assumed to follow a Geometric Brownian motion

$$dV_i = rV_i dt + \sigma_i V_i dW_i,\ i \in \{1, \ldots, m\}, \tag{1}$$

where r is the constant risk-free rate of return and σ_i the volatility parameter, and W_i is a Brownian motion. Each company is assumed to be funded by equity S_i as well as one bond with face value K_i and present value L_i. The approach of Merton (1974) uses V_i to pay off debt at maturity of the contract. If V_i is insufficient to repay the debt, the company defaults. Thus, at maturity both, equity S_i and the bond L_i can be viewed as derivatives on the value V_i of the firm's assets, and the payoffs at maturity time T can be described as follows:

$$\begin{aligned} L_i(V_i, T) &= \min(K_i, V_i(T)) \\ S_i(V_i, T) &= \max(V_i(T) - K_i, 0) \end{aligned} \tag{2}$$

The payoff of the bond at maturity time T is the minimum of the face value K_i of the bond and the firm value $V_i(T)$, i.e. when the company value $V_i(T)$

falls below the face value K_i, the default of the company i is triggered. In this case the value of the bond L_i is equal to V_i (see Merton (1974)). The payoff of the shares is identical to the payoff of a European call option on the firm value with strike price K_i. As the companies' values are assumed to follow a log normal distribution, $V_i(t) = V_i(0)e^{(r-\frac{\sigma_i^2}{2})t+\tilde{W}_i(t)}$, where $\tilde{W}_i(t) = \sigma_i W_i(t)$ is a Brownian motion, i.e. normally distributed with expectation 0 and variance $\sigma_i^2 t$. In our framework we assume that the covariance matrix follows a mean-reverting process. Next, we will apply Principal Component Analysis on $\tilde{W} = (\tilde{W}_1, \ldots, \tilde{W}_m)$ for dual purposes, to reduce the dimension implied by the m companies in the portfolio and at the same time to get some flexibility for adding some complexity in terms of stochastic volatility/correlation.

2.1 Reduction of the complexity: Principal Component Analysis

Principal Component Analysis (see Alexander (2000), (2001) and Joliffe (2002)) will allow one to find m uncorrelated variables at each time t, called the principal components of $\tilde{W}(t)$. Each principal component is a simple linear combination of the original returns as we can see below. Moreover, it is possible to state how much of the original variation in the data is explained by each principal component, which are ordered according to the amount of variation they explain. Now let the instantaneous stochastic covariance matrix $\Sigma(t)$ of $\tilde{W}(t)$, (defined as the quadratic variation of the process W, $\mathrm{d}\langle W(t)\rangle$) be decomposable as:

$$\Sigma(t) = AD(t)A', \tag{3}$$

where A is a $(m \times m)$ orthogonal (time independent) matrix ($A = (a_{j,i})$) with columns represented by the eigenvectors of $\Sigma(t)$ and $D(t)$ is a (time dependent) $(m \times m)$ diagonal matrix with the respective eigenvalues on its diagonal, which are assumed stochastic, i.e. $D(t) = diag(\lambda_1(t), \lambda_2(t), \ldots, \lambda_m(t))$. Therefore, the principal components of $\tilde{W}$ are obtained as:

$$B(t) = \tilde{W(t)}A \tag{4}$$

Hence, a linear transformation of the original risk factor returns has been made in such a way that transformed risk factors are orthogonal. More precisely, at this stage the principal components have zero instantaneous correlation. In the next proposition the instantaneous variances and correlations of $\tilde{W}$ and B are described:

Proposition 1. *The model exhibits stochastic volatility and correlation:*

$$\begin{aligned}
\sigma_{\tilde{W}_j}(dt) &= d\left\langle \tilde{W}_j, \tilde{W}_j \right\rangle^1 = \left[\sum_{i=1}^{m} a_{j,i}^2 \lambda_i(t)\right] dt \qquad (5)\\
\rho_{(\tilde{W}_j,\tilde{W}_k)}(t) &= \frac{d\left\langle \tilde{W}_j, \tilde{W}_k \right\rangle}{\sqrt{d\left\langle \tilde{W}_j, \tilde{W}_j \right\rangle d\left\langle \tilde{W}_k, \tilde{W}_k \right\rangle}} \qquad (6)\\
&= \frac{\sum_{i=1}^{m} a_{j,i} a_{k,i} \lambda_i(t)}{\sqrt{\sum_{i=1}^{m} a_{j,i}^2 \lambda_i(t)} \sqrt{\sum_{i=1}^{m} a_{k,i}^2 \lambda_i(t)}}
\end{aligned}$$

While:

$$\begin{aligned}
\sigma_{B_j}(dt) &= [\lambda_j(t)]\, dt \qquad (7)\\
\rho_{(B_j,B_k)}(t) &= \frac{0}{\sqrt{\lambda_j(t)}\sqrt{\lambda_k(t)}}
\end{aligned}$$

The implications of this model for the conditional dependence structure, $(\langle \tilde{W}_i(s), \tilde{W}_j(s) \rangle_{[t,T]})$ depend on the particular process impose on the eigenvalues. This would be completed in section 2.2.
The new risk factors are ordered by the amount of the stationary variation they explain. As A is a orthogonal matrix the relationship (4) is equivalent to $\tilde{W} = BA'$, i.e. $\tilde{W}_i(t) = a_{i1}B_1(t) + a_{i2}B_2(t) + \ldots + a_{im}B_m(t)$. Most of the time few eigenvectors are sufficient to describe more than 90% of the variation in the system (see Alexander (2000), (2001)), which allows us to reduce dimension of our system by removing the eigenvectors associated to low stationary volatilities. Thus, we set:

$$B^*(t) = \tilde{W}(t)A^*,\ B^* = (B_1(t), \ldots, B_{m_1}(t)) \qquad (8)$$

where A^* includes the first m_1 columns of A, i.e. it is a $m \times m_1$ orthogonal matrix. Therefore the kth columns of $B(t)$ have zero instantaneous correlation and their distribution given the process for $D(t)$ is normal $B_k(t)|D(t) \sim N(0, D(t))$. As we assume the instantaneous covariance matrix of the companies in the portfolio to be stochastic, then that of $B^*(t)$, $D(t) = diag(\lambda_1(t), \lambda_2(t))$ would also be stochastic because it is the result of

a linear transformation. In terms of the original data $\Sigma(t)$, $\tilde{W}_i(t)$ and $V_i(t)$, the transformation (8) is equivalent to

$$\begin{aligned} \Sigma(t) &= A^* D(t) A^{*'} + \Sigma_\epsilon(t) \\ \tilde{W}_i(t) &= a_{i1} B_1(t) + \ldots + a_{im_1} B_{m_1}(t) + \epsilon_i \\ \ln V_i(t) &= \mu_i(t) + \tilde{W}_i(t) + \ln V_i(0), \end{aligned} \tag{9}$$

where $\mu_i(t)$, the mean of the log returns of the company i, $i \in \{1, \ldots, m\}$ in t, is a function of the volatilities of the singular time steps. ϵ_i describes the noise caused by only including two eigenvectors and Σ_ϵ its variation. In the following we will ignore ϵ_i and therefore Σ_ϵ, the rationale for this is based on the observable fact that the remaining components have low stationary volatility and therefore they, in average, explain poorly the instantaneous volatility. The process for $B(t)$ will be described next.

2.2 Underlying Processes

We suggest the following model for the underlying system of principal components and eigenvectors k, j, where $k, j \in \{1, ..., m_1\}$:

$$dB_k = \sqrt{\lambda_k} dZ_k \tag{10}$$

$$d\lambda_k = \beta(\lambda_k)dt + \gamma \lambda_k dQ_k \tag{11}$$

$$\begin{aligned} E(dZ_k dQ_k) &= 0 \\ E(dZ_k dZ_j) &= 0 \quad \forall\ k \neq j \\ E(dQ_k dQ_j) &= 0, \end{aligned} \tag{12}$$

Where 11 satisfies the standard assumptions for existence, stationary and mixing conditions (see Genon-Catalot et. al. 2000). In particular, due to its consistency and popularity for modelling volatilities, we will work with a particular case of 13:

$$d\lambda_k = d_k(b_k - \lambda_k)dt + c_k \sqrt{\lambda_k} dQ_k \tag{13}$$

here d_k, b_k and c_k are fixed constants, λ_k is the kth eigenvalue and dZ_k and dQ_k are independent Wiener processes[2]. These processes are presented under the risk neutral measure, therefore the parameter d_k has been already corrected to account for the change of measure.

[2]The processes for $B_k(t)$ and $d\lambda_k$ are known to exist. The original data $\tilde{W}_i(t)$, $S_i(t)$, $i \in \{1, \ldots, m\}$ and $\Sigma(t)$ can be expressed as a linear transformation of $B_k(t)$, $D_k(t)$ and A (Equations 4-6). Therefore the processes for W_i, V_i and $\Sigma(t)$ exist.

This is a continuous time stochastic volatility model (see Genon-Catalot (2000) for unidimensional process) for the principal components. The components' eigenvalues, which represent their volatility, follow the popular Cox-Ingersoll-Ross (1979) model (also known by the work of Heston (1993)).

Remark 1. *Some general clarifications:*

1. *Equations 1 to 12 lead to a multidimensional stochastic volatility model for the security prices.*

2. *From a theoretical viewpoint (see Bjork 1998), if $m_1 < \frac{m}{2}$ then this system could lead to arbitrage opportunities. Nevertheless, from a practical side, those opportunities should be insignificant as long as the error ϵ_i is small.*

The settings 1 to 12, particularly equations 10-12 for the eigenvalue process, have an important implication for the dependence structure in time and space of the principal components B. Using the Kolmogorov-forward partial differential equation (PDE) to compute the joint density function, it can be shown that this PDE is separable. Therefore the distribution at time T for B_j conditional on t ($t < T$) is independent of that of B_i, for all $i \neq j$. In other words, the principal components are not only instantaneously independent by construction but also conditional independent by means of the process selected for the eigenvalues.

3 Tree Development

Considering expression (5) for $t = 0$,

$$B^*(0) = \tilde{W}(0)A^*, \tag{14}$$

we are going to build a combined tree model for the stochastic volatility and the underlying B^* to get values and probabilities for B^* and D at maturity T. As seen above, these results can be expressed in terms of the original variables $\tilde{W}_i(T)$ and $V_i(T)$, $i \in \{1, \ldots, m\}$ as defined by (6-8).
This data allow us to calculate how many companies default, assuming we know the face values K_i of the bonds and the respective probabilities, i.e. all the information needed to compute the value of the tranches of a CDO, without having lost much accuracy.

3.1 Trinomial-Tree Approximation for the Transform of the Brownian motion

For simplicity of explanation, we set $m_1 = 2$ from this section on. The process (10) for $dB_k,\ k \in \{1, 2\}$, is approximated using a trinomial-tree on the lines of Rubinstein (2000) because in this framework the spacing parameter can be set independent from the volatility of the process. In the following, the nodes are denoted by (j, t), where j is the number of upwards movements and t indicates the number of time steps passed since $t = 0$, i.e. the root of the tree. Let $B_k(t) = B_k(0) + j\Delta B_k$, then

$$\Delta B_k(t) = B_k(t+1) - B_k(t) = \begin{cases} \Delta B_k & \text{with probability} \quad p_{j,j+1,t} \\ 0 & \text{with probability} \quad p_{j,j,t} \\ -\Delta B_k & \text{with probability} \quad p_{j,j-1,t} \end{cases} \tag{15}$$

The convergence of this discrete three-point distribution to the continuous process is ensured by matching the first two moments. Note that $E(B_k(0)) = 0$. Again, let $B_k(t) = B_k(0) + j\Delta B_k = j\Delta B_k$. Then,

$$\begin{aligned} E(\Delta B_k(t)) = p_{j,j+1,t}\Delta B_k - p_{j,j-1,t}\Delta B_k &= 0 && (16) \\ Var(\Delta B_k(t)) = E(\Delta B_k^2(t)) &= p_{j,j+1,t}\Delta B_k^2 + p_{j,j-1,t}\Delta B_k^2 \\ &= \lambda_k(j\Delta B_k, (t-1)\Delta t)\Delta t && (17) \\ p_{j,j+1,t} + p_{j,j,t} + p_{j,j-1,t} &= 1 && (18) \end{aligned}$$

From Equation (16) follows

$$p_{j,j+1,t} = p_{j,j-1,t} \tag{19}$$

Substituting Equation (19) in Equation (17) we get

$$\Delta B_k^2(2p_{j,j+1,t}) = \lambda_k(j\Delta B_k, (t-1)\Delta t)\Delta t \tag{20}$$

Solving this for $p_{j,j+1,t}$ we obtain

$$p_{j,j+1,t} = \frac{1}{2}\frac{\lambda_k(j\Delta B_k, (t-1)\Delta t)}{\Delta B_k^2}\Delta t \tag{21}$$

and

$$\begin{aligned} p_{j,j-1,t} &= \frac{1}{2}\frac{\lambda_k(j\Delta B_k, (t-1)\Delta t)}{\Delta B_k^2}\Delta t && (22) \\ p_{j,j,t} &= 1 - \frac{\lambda_k(j\Delta B_k, (t-1)\Delta t)}{\Delta B_k^2}\Delta t, && (23) \end{aligned}$$

To guarantee positive probabilities $p_{j,j+1,t}, p_{j,j,t}$ and $p_{j,j-1,t}$ we choose appropriate spacing and timing parameters. From equations (21), (22) and (23) it can be shown that the probabilities are positive if

$$\lambda_k(j\Delta B_k, (t-1)\Delta t) < \frac{\Delta B_k^2}{\Delta t}. \tag{24}$$

is satisfied (see Escobar et al. (2007)). Increasing the number of time steps, Hull and White (1990) and Brennan and Schwartz (1978) find it desirable to keep the ratio $\frac{\Delta B_k^2}{\Delta t}$ constant to ensure convergence. In order to develop a recombining tree with a spacing parameter not depending on the volatility, i.e. the eigenvalue, the eigenvalue has to be restricted. However, in order not to restrain the model too much we use the following relationship between the variances of the original data and the eigenvalues

$$\begin{aligned}\sum_{i=1}^{m} Var(V_i(t)) = trace(\Sigma(t)) &= trace(\Sigma(t)AA') \\ &= trace(A'\Sigma(t)A) = trace(D(t)) = \sum_{k=1}^{m} \lambda_k(t)\end{aligned}$$

(see Joliffe (2002)). Thus, we set

$$\Delta B_k^2 \equiv \gamma \Delta t \sum_{k=1}^{m} \lambda_k(0) \tag{25}$$

and inequality (24) transforms to

$$\lambda_k(j\Delta B_k, (t-1)\Delta t) < \gamma \sum_{k=1}^{m} \lambda_k(0). \tag{26}$$

A sufficient choice of γ is possible as all $\lambda_k(t)$ are bounded in the trinomial tree. The tree building and the combination of the trees are shown for the first two principal components. Table (1) illustrates the possible movements of B_1 and B_2 after two time steps. The probabilities of each joint combination are obtained by simply multiplying the probabilities of the marginals as the marginals are uncorrelated. These marginal probabilities are, however, influenced by the volatilities $\sqrt{\lambda_1(t-1)}$ and $\sqrt{\lambda_2(t-1)}$ respectively.

Table 1: Possible Movements of $B_1(t)$ and $B_2(t)$ after two time steps

$2\Delta B_1(t)$ $2\Delta B_2(t)$	$2\Delta B_1(t)$ $\Delta B_2(t)$	$2\Delta B_1(t)$ 0	$2\Delta B_1(t)$ $-\Delta B_2(t)$	$2\Delta B_1(t)$ $-2\Delta B_2(t)$
$\Delta B_1(t)$ $2\Delta B_2(t)$	$\Delta B_1(t)$ $\Delta B_2(t)$	$\Delta B_1(t)$ 0	$\Delta B_1(t)$ $-\Delta B_2(t)$	$\Delta B_1(t)$ $-2\Delta B_2(t)$
0 $2\Delta B_2(t)$	0 $\Delta B_2(t)$	0 0	0 $-\Delta B_2(t)$	0 $-2\Delta B_2(t)$
$-\Delta B_1(t)$ $2\Delta B_2(t)$	$-\Delta B_1(t)$ $\Delta B_2(t)$	$-\Delta B_1(t)$ 0	$-\Delta B_1(t)$ $-\Delta B_2(t)$	$-\Delta B_1(t)$ $-2\Delta B_2(t)$
$-2\Delta B_1(t)$ $2\Delta B_2(t)$	$-2\Delta B_1(t)$ $\Delta B_2(t)$	$-2\Delta B_1(t)$ 0	$-2\Delta B_1(t)$ $-\Delta B_2(t)$	$-2\Delta B_1(t)$ $-2\Delta B_2(t)$

3.2 Trinomial-Tree for the Eigenvalue Matrix

The process (13) is implemented using a trinomial-tree suggested by Hull and White (1990). This method requires a constant volatility parameter for the process, which can be obtained with the following transformation of the process:

$$\phi_k = \sqrt{\lambda_k},\ k \in \{1, 2\} \tag{27}$$

Using Itô's Lemma, this leads to

$$\begin{aligned} d\phi_k &= d\sqrt{\lambda_k} = (\frac{1}{2\sqrt{\lambda_k}}(b_k d_k - d_k\lambda_k) - \frac{1}{8}\lambda_k^{-\frac{3}{2}}c_k^2\lambda_k)dt + \frac{c_k}{2}dQ_k \\ &= (\frac{b_k d_k}{2\phi_k} - \frac{d_k}{2}\phi_k - \frac{1}{8}c_k^2\frac{1}{\phi_k})dt + \frac{c_k}{2}dQ_k \\ &= (\frac{4b_k d_k - c_k^2}{8}\frac{1}{\phi_k} - \frac{d_k}{2}\phi_k)dt + \frac{c_k}{2}dQ_k \\ &= q_k dt + \nu_k dQ_k, \end{aligned}$$

where $\nu_k = \frac{c_k}{2}$, $q_k = \frac{\alpha_{1,k}}{\phi_k} - \alpha_{2,k}\phi_k$ and $\alpha_{1,k} = \frac{4d_k b_k - c_k^2}{8}, \alpha_{2,k} = \frac{d_k}{2}$.
Again the lifetime of the derivative is divided in $n = \frac{T}{\Delta t}$ equal time steps, where Δt is the length of one such time step. After each time step the tree branches out. The nodes are denoted by (l, t), where l is the number of upwards movements of the eigenvalue, i.e. the value $\phi_k(l, t) = \phi_k(0) + l\Delta\lambda$, and t indicates the number of time steps passed since $t = 0$. As before, we assume that λ_k can increase, move aside, or decrease. However, as q is unbounded, the model we used before may not converge. Thus, this method

was modified by Hull and White (1990). Beside the possibility that $\Delta\phi_k$ is equal to $\Delta\phi_k(l,t), 0$ and $-\Delta\phi_k(l,t)$ two other branching methods are relevant for this implementation. The three methods are illustrated in Figures 1-3, where κ is the move increment.

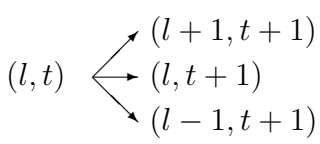

Figure 1: Relation between time step t and $t+1$ when $\kappa = l$

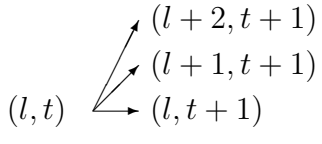

Figure 2: Relation between time step t and $t+1$ when $\kappa = l+1$

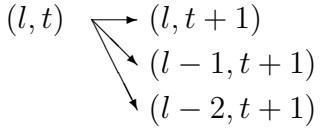

Figure 3: Relation between time step t and $t+1$ when $\kappa = l-1$

To ensure convergence the probabilities are set to (see Escobar et al. (2007)):

$$p_{l,\kappa+1} = \frac{\nu_k^2\Delta t}{2(\Delta\phi_k)^2} + \frac{\eta_k^2}{2(\Delta\phi_k)^2} + \frac{\eta_k}{2\Delta\phi_k} \tag{28}$$

$$p_{l,\kappa} = 1 - \frac{\nu_k^2\Delta t}{(\Delta\phi_k)^2} - \frac{\eta_k^2}{(\Delta\phi_k)^2} \tag{29}$$

$$p_{l,\kappa-1} = \frac{\nu_k^2\Delta t}{2(\Delta\phi_k)^2} + \frac{\eta_k^2}{2(\Delta\phi_k)^2} - \frac{\eta_k}{2\Delta\phi_k}, \tag{30}$$

where

$$\eta = q_k(l,t)\Delta t + (l-\kappa)\Delta\phi_k$$

When $\Delta\phi_k$ is set to $\nu_k\sqrt{3\Delta t}$ the following dynamic rules for the choice of κ can be implemented to ensure positive probabilities:

$$\kappa = \begin{cases} l+1 & \text{if} \quad \frac{q_k(l,t)\Delta t}{\Delta\phi} \geq \sqrt{\frac{2}{3}} \\ l & \text{if} \quad -\sqrt{\frac{2}{3}} < \frac{q_k(l,t)\Delta t}{\Delta\phi} < \sqrt{\frac{2}{3}} \\ l-1 & \text{if} \quad \frac{q_k(l,t)\Delta t}{\Delta\phi} \leq -\sqrt{\frac{2}{3}} \end{cases} \tag{31}$$

These dynamic rules of choice for κ imply minimum and maximum values for $\phi_k(t)$:

$$-\sqrt{\tfrac{2}{3}} \leq \tfrac{q_k(l,t)\Delta t}{\Delta\phi} \leq \sqrt{\tfrac{2}{3}} \;\Leftrightarrow\; \phi_{k,min} = \tfrac{-\beta+\sqrt{\beta^2+4\alpha_1\alpha_2}}{2\alpha_2} \leq \phi_k(t) \leq \tfrac{\beta+\sqrt{\beta^2+4\alpha_1\alpha_2}}{2\alpha_2} = \phi_{k,max} \tag{32}$$

where $\beta = \sqrt{\frac{2}{3}}\frac{\Delta\phi_k}{\Delta t}, \alpha_1 = \frac{4db-c^2}{8}, \alpha_2 = \frac{1}{2}d$.
The branching method is changed to $\kappa = l-1$ at a node (ψ, t), where ψ is the largest integer that $\phi_k(\psi, t) = \phi_k(0)+\psi\Delta\phi \leq \phi_{k,max}$ and to $\kappa = l+1$ at a node (ζ, t), where ζ is the smallest integer that $\phi_k(\zeta, t) = \phi_k(0) + \zeta\Delta\phi \geq \phi_{k,min}$.

3.3 Combined Tree for Principal Components and Eigenvalues

In order to obtain spacing parameters, independent from the volatility of the principal components trees, we had to restrict the volatility (see (26)) generated by the trees for the eigenvalues, $\sqrt{\lambda_k},\ k \in \{1, 2\}$. Thus, additionally to the constraints on the volatility imposed by (32) the following equation has to hold:

$$\phi_k(j\Delta B_k, (t-1)\Delta t) < \sqrt{\gamma \sum_{k=1}^{m} \lambda_k(0)} \tag{33}$$

Hence, as soon as one of the constraints (32) and (33) is reached, in a time step t, it causes the tree for ϕ_k to change the branching method in this node (s, t). Here s is $\min(\psi, \vartheta)$, where ϑ is the largest integer such that $\phi_k(\vartheta, t) = \phi_k(0) + \vartheta\Delta\phi \leq \sqrt{\gamma \sum_{k=1}^{m} \lambda_k(0)}$ in (33) while ψ is the largest integer that $\phi_k(\psi, t) = \phi_k(0) + \psi\Delta\phi \leq \phi_{k,max}$ in (32).
As the Brownian motions of the principal components of the assets and of the eigenvalues are independent, then the joint probabilities can be obtained by simply multiplying the marginal probabilities. The probabilities for the movements of the principal components ((21) to (23)) also apply in the case of stochastic volatilities.
The nodes in the combined tree are denoted by (j, w, l, v, t), where j and w indicate the number of up or down moves of the first and the second principal component respectively. l and v specify the level of the two eigenvalues. The trees of the eigenvalues and the principal components are arranged in series, i.e. the eigenvalues in t influence the probabilities of the principal components to move up or down in $t + 1$ (in the tree the principal components increase or decrease n times, the eigenvalues $n - 1$ times until T). A particular node in the combined tree branches in 3^4 different nodes in the next time step.

3.4 Computation of the Means

In T, $\mu_i(T),\ i \in \{1,\ldots,m\}$, has to be added to the retransformed Brownian motions $\tilde{W}_i(T)$. $\mu_i(T)$ is actually the sum of the means of the singular time steps and can be obtained by combining the two trees of the eigenvalues to a tree for the variances of the underlyings with nodes (l,v,t) to $(3)^{(n-1)}$ nodes in T, assuming n time steps:

$$\mu_i(l,v,t|l,v,t-1;\ldots;l,v,t=0) = \sum_{t=1}^{n}(r - \frac{\sigma_i(l,v,t)^2}{2})\Delta t, \quad \forall \quad \text{combinations} \quad l,v,$$

where $\sigma_i(l,v,t)^2$ is computed by

$$\lambda_1(l,t)a_{1i}^{*2} + \lambda_2(v,t)a_{2i}^{*2} \quad \forall \quad \text{combinations} \quad l,v. \tag{34}$$

However, the determination of the exact means would result in a path-dependent pricing as $\mu_i(T)$ depends on the path in the combined volatility tree. This results in the computation of $3^{2\cdot(n-1)}$ different paths, where n is the number of time steps computed in the tree. The distribution of the values of the means in T is illustrated for an example in section 5.
We will compute the mean at each node (l,v,t) in the variance tree by weighting the means of the predecessors nodes with the conditional probabilities to reach the respective node (l,v,t). In this way each node of the combined variance tree in time step n gives rise to one mean value $\mu_i(T)$, i.e. there are $((n-1)\cdot 2+1)^2$ different mean values in T. This is illustrated in Figure 4 for $n=2$. In Section 5 we compare the distribution of the values resulting from this method with the path-dependent computation method. This comparison shows that the distributions are similar.
These mean values allow us to express the principal components and eigenvalues in T in terms of $\ln V_i$ and V_i, $i \in \{1,\ldots,m\}$:

$$\begin{aligned} \Sigma(T) &\approx A^* D(T) A^{*'} \\ \ln V_i(T) &\approx \ln V_i(0) + \mu_i(T) + \tilde{W}_i(T), \\ V_i(T) &\approx V_i(0)\exp\left(\mu_i(T) + \tilde{W}_i(T)\right) \end{aligned} \tag{35}$$

By subtracting the face values D_i from V_i it can be determined if and how many companies default in each scenario. The pay-off function of each tranche of the CDO can then be weighted with the joint probabilities of the respective scenario (i,j,l,v,T) and discounted, which results in the value of the tranche in $t=0$.

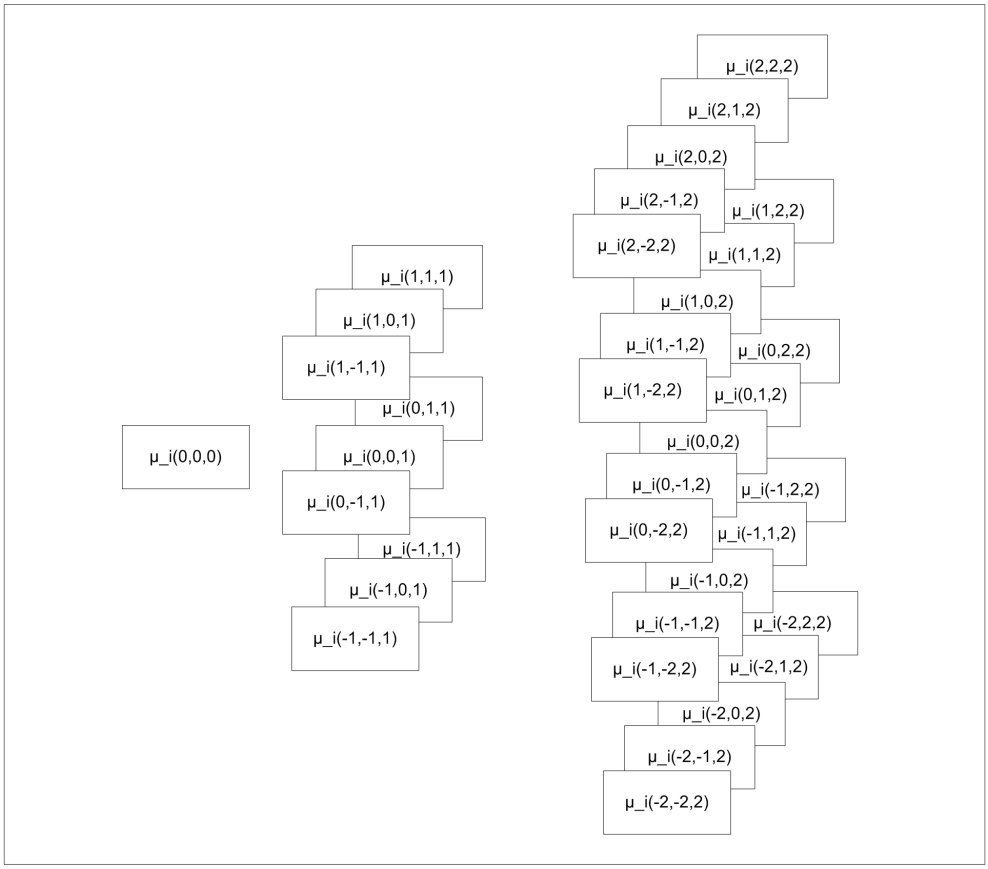

Figure 4: Tree for the Mean Values for $n = 2$ and two eigenvalues

4 Parameter Estimation

Estimation of multidimensional stochastic processes has been a silent topic for decades, where most of the work has been dedicated to the unidimensional case. The curse of dimensionality makes this topic specially challenging and nowadays the literature sees growth firstly for discrete time models and most recently also in their continuous time counterpart.

The estimation procedure to calibrate the parameters of our model is inspired by the work of Genon-Catalot et al. (2000) for stochastic volatility models in the case of a fixed sampling interval Δt. Our method is based on the idea of multidimensional Hidden Markov models. The reason for selecting this

method lies in its consistency, speed of calculation and simplicity, in future studies this could be the initial value in a more complex time consuming estimation method (see Sørensen (2000) for a survey).
It is important to realize that the data could result from two sources either companies with observable asset values, for which the estimation will be directly applied, or companies with only observable equity prices. For the second case, we rely on the assumption that the volatility structure of assets and equity is the same. This holds true, for example, in the case of companies with a relatively small debt (leverage) proportion.
In this section, we will first explain the relation between Hidden Markov Models and stochastic volatility (section 7.1), then this concepts will be applied for our multidimensional stochastic volatility model (section 7.2).

4.1 Stochastic volatility models as hidden Markov models

Genon-Catalot et al. (2000) show that a stochastic volatility model can be viewed as a hidden Markov model. See Definition 3.1 in Genon-Catalot et al. (2000) for the definition of a Hidden Markov Model.
Now consider a Markov process which is defined by a stochastic equation

$$dX = \mu(X)dt + \sigma(X)dZ, \quad X_0 = \eta, \tag{36}$$

where W is a standard Brownian motion and η is a real random variable defined on Ω and independent of Z. Genon et al. (2000) make the standard assumptions on functions $b(x)$ and $a(x)$, ensuring that the solution (see Assumption 1) of (36) is a positive recurrent diffusion on an interval (l, r), $-\infty \leq l < r \leq \infty$, (see Assumption 2) and a strictly stationary ergodic time-reversible process (see Assumption 3 in the referred paper).

Assumption 1.
The functions $\mu(x)$ and $\sigma(x)$ are defined on (l, r), and satisfy

$$\mu(x) \in C^1(l, r), \sigma^2(x) \in C^2(l, r), \sigma(x) > 0 \ \forall x \in (l, r), \tag{37}$$

and

$$\exists K > 0, \forall x \in (l, r), |\mu(x)| \leq K(1 + |x|) \text{ and } \sigma^2(x) \leq K(1 + x^2). \tag{38}$$

For $x_0 \in (l, r)$, define the scale and speed densities of diffusion (X_t),

$$s(x) = \exp(-2\int_{x_0}^{x} \frac{\mu(u)}{\sigma^2(u)}du), \quad m(x) = \frac{1}{\sigma^2(x)s(x)}. \tag{39}$$

Assumption 2.

$$\int_{l}^{\infty} s(x)dx = \infty, \int_{-\infty}^{r} s(x)dx = \infty, \int_{l}^{r} m(x)dx = M < \infty. \tag{40}$$

Let us define the stationary density

$$\pi^*(x) = \frac{1}{M}m(x)1_{x\in(l,r)}. \tag{41}$$

Assumption 3.
The initial random variable η has distribution $\pi^*(dx) = \pi^*(x)dx$.

Now let $(Y, V)_{t\geq 0}$ be a two-dimensional diffusion process given by

$$dY = \sigma dZ, Y(0) = 0, \tag{42}$$

$$V = \sigma^2, dV = d(b - V)dt + c\sqrt{V}dQ, V(0) = \eta, \tag{43}$$

where η is a random variable, independent of (Z, Q), and assume that Assumptions 1 to 3 hold for $(V)_{t\geq 0}$. Then, the diffusions V and Y is strictly stationary, ergodic and time-reversible (see Proposition 3.2 in Genon-Catalot et al. (2000)).
These features are used to create moments-like estimators. For positive Δt we define, for $i \geq 1$,

$$S(i) = \frac{1}{\sqrt{\Delta t}} \int_{(i-1)\Delta t}^{i\Delta t} \sigma_s dZ_s \tag{44}$$

and

$$U(i) = (\overline{V}(i), V(\Delta t)), \quad \overline{V}(i) = \frac{1}{\Delta t} \int_{(i-1)\Delta t}^{i\Delta t} V_s ds, \tag{45}$$

where $V(\Delta t)$ are values of V at fixed intervals Δt.

Assume that the above hidden diffusion V satisfies the Assumptions 1 to 3 and that $E(V(0)^2)$ is finite. Then, $E(\overline{V}(1)) = E(\overline{V}(0)) = \beta$, and

$$E(\overline{V}(1)^2) = b^2 + \frac{c^2 b}{2d}\frac{2(d\Delta t - 1 + e^{-d\Delta t})}{d^2(\Delta t)^2}, \quad (46)$$

$$E(\overline{V}(1)\overline{V}(2)) = b^2 + \frac{c^2}{2d}b\frac{(1 - e^{-d\Delta t})^2}{d^2(\Delta t)^2}. \quad (47)$$

Moreover, the following functions of the observations

$$\widehat{m}_1 = \frac{1}{n}\sum_{i=1}^{n} Z(i)^2, \quad \widehat{m}_2 = \frac{1}{3n}\sum_{i=1}^{n} Z(i)^4, \quad \widehat{m}_{12} = \frac{1}{n}\sum_{i=1}^{n-1} Z(i)^2 Z(i+1)^2, \quad (48)$$

where n is the number of observations, are consistent estimators of b^2, $E(\overline{V}(1)^2)$ and $E(\overline{V}(1)\overline{V}(2))$ respectively. For a proof see Genon-Catalot et al. (2000).

4.2 Application to the estimation of the eigenvalue parameters

In our multidimensional model, an orthogonal transformation of the returns leads to a set of univariate independent stochastic volatility processes. Therefore we study the idea of extending univariate estimations methods as the one explain before to get our process.
For our estimation, we propose a two steps procedure. First, the constant orthogonal matrix A^* is found by singular decomposition on the stationary covariance structure. Second, we use the procedure outlined in the previous section to estimate the parameters in each principal component (SV process). In a separate paper, the details on how the stationarity and ergodicity of the principal components implied the same attributes on the observable increments of the multidimensional log-prices will be provided. These features are necessary for our estimators to be consistent.
In the first step, we consider a diffusion vector process $V(t)$, which is observable at n discrete times with regular sampling interval Δt. Let us denote the vector of observations by $V(\Delta t)$, the vector of returns would be $R(\Delta t) = \frac{V(\Delta t) - V(\Delta(t-1))}{V(\Delta(t-1))}$. The covariance matrix of $R(\Delta t)$ is found using standard

methods, Σ_1. A^* is taken as the orthogonal matrix in a singular decomposition of Σ_1.
Second step, we consider a diffusion process $(B(t), D(t))$, where we regard $B(t)$, the matrix of the principal components, as observable[3] at n discrete times with regular sampling interval Δt and denote the observations by $B(\Delta t)$. $D(t)$, the matrix of the eigenvalues, is assumed to be ergodic (see conditions on previous section) and rules the diffusion of $(B(\Delta t))$. The observations $B(t)$ can be viewed as a hidden Markov model. For Δt positive, we define, for $i \geq 1$,

$$B(i) = \frac{1}{\sqrt{\Delta t}} \int_{(i-1)\Delta t}^{i\Delta t} \sqrt{\lambda(s)} dZ \tag{49}$$

and

$$\overline{D}(i) = \frac{1}{\Delta t} \int_{(i-1)\Delta t}^{i\Delta t} D(s) ds. \tag{50}$$

We subtract in each sampling interval the mean $\overline{V}_i$ from all data series V_i to obtain m data sets whose mean is zero. For these m sets we compute the first two principal components to obtain two-dimensional $B(i)$.
Thus, for our model (10 - 12) we can apply that $E(\overline{D}(1)) = E(\overline{D}(0)) = b$, and

$$E(\overline{D}(1)^2) = b^2 + \tfrac{c^2}{2d} b \tfrac{2(d\Delta t - 1 + e^{-d\Delta t})}{d^2(\Delta t)^2}, \quad E(\overline{D}(1)\overline{D}(2)) = b^2 + \tfrac{c^2}{2d} b \tfrac{(1-e^{-d\Delta t})^2}{d^2(\Delta t)^2}. \tag{51}$$

Consistent estimators of b, $E(\overline{D}(1)^2)$ and $E(\overline{D}(1)\overline{D}(2))$ are the following functions of the observations

$$\widehat{m}_1 = \tfrac{1}{n} \sum_{i=1}^{n} Z(i)^2, \quad \widehat{m}_2 = \tfrac{1}{3n} \sum_{i=1}^{n} Z(i)^4, \quad \widehat{m}_{12} = \tfrac{1}{n} \sum_{i=1}^{n-1} Z(i)^2 Z(i+1)^2. \tag{52}$$

which allow us to estimate the parameters d, b, c for the diffusions of the first two eigenvalues.
As mentioned before, the literature on estimation methods for continuous-time multidimensional stochastic volatility processes is almost non-existent.

[3] $B(t)$ is a linear transformation of the observable data. The fact that A^* is also calibrated leads to less efficient still consistent estimators.

This method, like any other moment-based procedure, has several difficulties for small and medium sample sizes as well as for some ranges of the parameters (see Sørensen (2000) for a survey). We have, however, already tested the method on simulations, i.e. we tried to retrieve the parameters of the simulations using our parameter estimation method. This showed that the results for the mean-reversion level and the mean-reversion speed were quite close. On the other hand, its benefit lies in the speed of calibration, comparing to other proposed methods like Indirect Inference, Simulated Maximum Likelihood or Markov Chain Monte Carlo, which first depends on simulations, secondly, based on one dimensional experimentation (see Sørensen (2000)) are slower than our.

5 Empirical Results and CDO Example

We start this section by showing empirical evidence of few nonzero principal components. Then an application on CDOs is presented.
The following graph shows the explanatory power of each principal component on the returns from the companies in SP100, for the period 1983-2006. Less than 7 components explain 97% of the total variance.

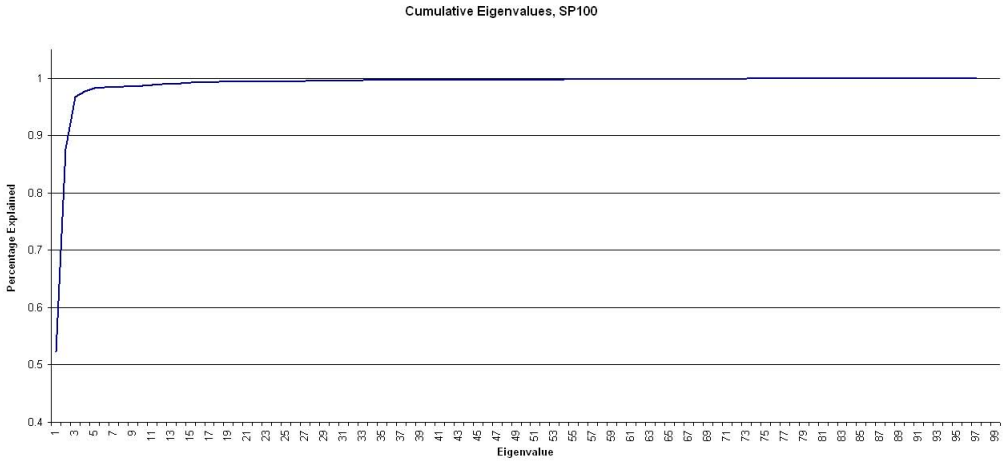

Figure 5: Cumulative Eigenvalue Explained

This is evidence of the benefits of working with few principal components.

In order to assess the impact of the stochastic principal components on derivatives, we select the pricing of a CDO. In the remaining part of this section, the tranches of a real-data based CDO under the stochastic PCA model will be priced. In a second part we assess the prices under a standard Gaussian model (constant eigenvalues) as well as the speed of convergence depending on the volatility explained by the eigenvalues.
The underlying security is a portfolio of coupon paying corporate bonds on 10 firms. We assume the face values $K_i, i \in \{1, \ldots, 10\}$ of these bonds to be 1 and assume that the process of the company values, $V(t)$, show the same characteristics as the respective observable stock price processes, $S(t)$ (based on Merton's framework, this applies either to low debt companies or those for which no information on assets is available). Thus, in order to estimate the parameters of the eigenvalue process we use the stock returns of ten companies (MMM.N, ABT.N, AA.N, AXP.N, BUD.N, AVP.N, BAX.N, BDK.N, BNI.N, BMY.N)[4] from 1983 to October 2006. For these companies, no information regarding assets values or debt is available, only probabilities of default are known, these are used to have an idea on default thresholds for stock prices. The following table suggest the number of components to choose as 2:
We use the parameter estimation method described above for the mean-reversion level and the mean-reversion speed. As the method has still to be improved, considering the volatilities of the eigenvalue, we assume plausible values for the latter. Thus, we value the tranches for the following scenario:

Basic Scenario: $r = 0.05$, Maturity = 1 year, $\Delta t = \frac{1}{n}$, $n = 3$, $b_1 = 0.2713032$, $b_2 = 0.126441$, $d_1 = d_2 = 0.5$, $c_1 = 0.14$, $c_2 = 0.12$, $K_i = 0.75 \cdot V_i(0)$

Basic Scenario: Tranche 1 = from 0 to 1 default, Tranche 2 = from 2 to 3 defaults, Tranche 3 = from 3 to 4 defaults, Tranche 4 = from 4 to 5 defaults, Tranche 5 = 5 to 10 defaults

We value the tranches using the path-dependent and the non path-dependent

[4] 3M Common Stock , Abbott Labs Common Stock, Alcoa Inc Common Stock, Anheuser Busch Common Stock, Avon Prods Common Stock , Baxter Intl Common Stock , Black and Decker Common Stock , BNSF Common Stock and Bristol Myers Common Stock respectively

Table 2: Volatility explained by the Eigenvalues

Number of Eigenvectors	Percentage of the Volatility explained
1	66.045%
2	81.24%
3	85.824%
4	89.6124 %
5	92.9287 %
6	95.1642 %
7	96.9177 %
8	98.472 %
9	99.4742%
10	99.99929%

mean value computation in order to compare the distributions we get for the mean values in T and the values of the tranches. The results for the values of the tranches of these two methods are close (see Table 8).

Table 3: Values of the Tranches using the path dependent method

Tranche	path-dependent method	Non path-dependent method
1	0.11409	0.11490
2	0.07451	0.07451
3	0.07275	0.07275
4	0.05431	0.05431
5	0.03777	0.03766

Furthermore, we analyze the distribution of the means of the two methods. Using the Mann-Whitney Test (as well as the Kolmogorov-Smirnov, see Lehmann (2006)) we cannot reject the H_0 hypothesis that the two distributions are alike. The two distributions are shown in the appendix. and analyzed in Table 4. However, the values of the tranches differ more when we assume higher volatilities for the process of the eigenvalues. In the following we have calculated the value for the volatilities c_1=0.25 and c_2=0.13 (see Table 5).
The deviation in value of the first tranche can be explained by the fact that

Table 4: Moments of the Mean Values

	path-dependent Method	Non path-dependent Method
Mean	0.026241208	0.026125472
Variance	$4.92252E-06$	$5.82982E-06$
Skewness	-0.242396164	-0.195523145
Minimum	0.018953	0.018953
Maximum	0.031259	0.031259

Table 5: Values of the Tranches using the path-dependent method with higher volatilities

Tranche	path-dependent method	Non path-dependent method
1	0.1291563212	0.1423252046
2	0.0745361782	0.0745268594
3	0.0727739773	0.0727548047
4	0.0544242608	0.0544113424
4	0.0392558059	0.0544113424

the differences in skewness between the distributions of these two methods become higher for higher volatilities of the eigenvalue process although the span of both distributions, i.e. minimum and maximum values, remains the same for both distributions (see Tables 4 and 6). This imprecision can be handled by using more time steps for the computation of the tranches when the processes of the eigenvalues are quite volatile. The two distributions are shown in the appendix and analyzed in Tables 6.

Table 6: Moments of the Mean Values

	path-dependent Method	Non path-dependent Method
Mean	0.026235437	0.026026984
Variance	$1.82868E-05$	$1.54905E-05$
Skewness	-0.432561011	-0.330387539
Minimum	0.013258	0.013258
Maximum	0.034434	0.034434

We show next the difference in prices of the various tranches, when comparing to a Gaussian framework. Gaussianity is a particular case of our framework achieved when volatilities and speed of reversion for the eigenvalues are as-

sumed zero and initial value is equated to the mean reversion level. This is the constant eigenvalue framework or static principal component analysis. The setting is as before:

Basic Scenario: $r = 0.05$, Maturity $= 1$ year, $\Delta t = \frac{1}{n}$, $n = 3$, $b_1 = 0.2713032$, $b_2 = 0.126441$, $d_1 = d_2 = 0$, $c_1 = c_2 = 0$, $K_i = 0.75 \cdot V_i(0)$

The tranches' price is provided in tables 7 and 8. Table 8 allows for higher eigenvalues: $b_1 = 0.47130$, $b_2 = 0.3264$:

Table 7: Values of the Tranches using the path dependent method

Tranche	path-dependent method
1	0.104438
2	0.07186
3	0.07054
4	0.054238
5	0.039181

Table 8: Values of the Tranches using the path dependent method

Tranche	path-dependent method
1	0.10451
2	0.07191
3	0.07059
4	0.05426
5	0.03919

Those results show the differences between static principal component (Gaussian framework) and SPCA approaches along all tranches. Note that the difference in prices could be as much as 25%.
The following figure 6 give some light on the speed of convergence of the tree approach as well as on the need for a larger number of eigenvalues. It pictures the price of a tranche as the number time steps and volatility explained increases. The graph shows that first the convergence is quite fast regardless of the level of volatility explained (as long as the explanatory power is above a reasonable threshold, i.e. 80%) and secondly the difference in prices is minimal if the volatility explained is above the same threshold.

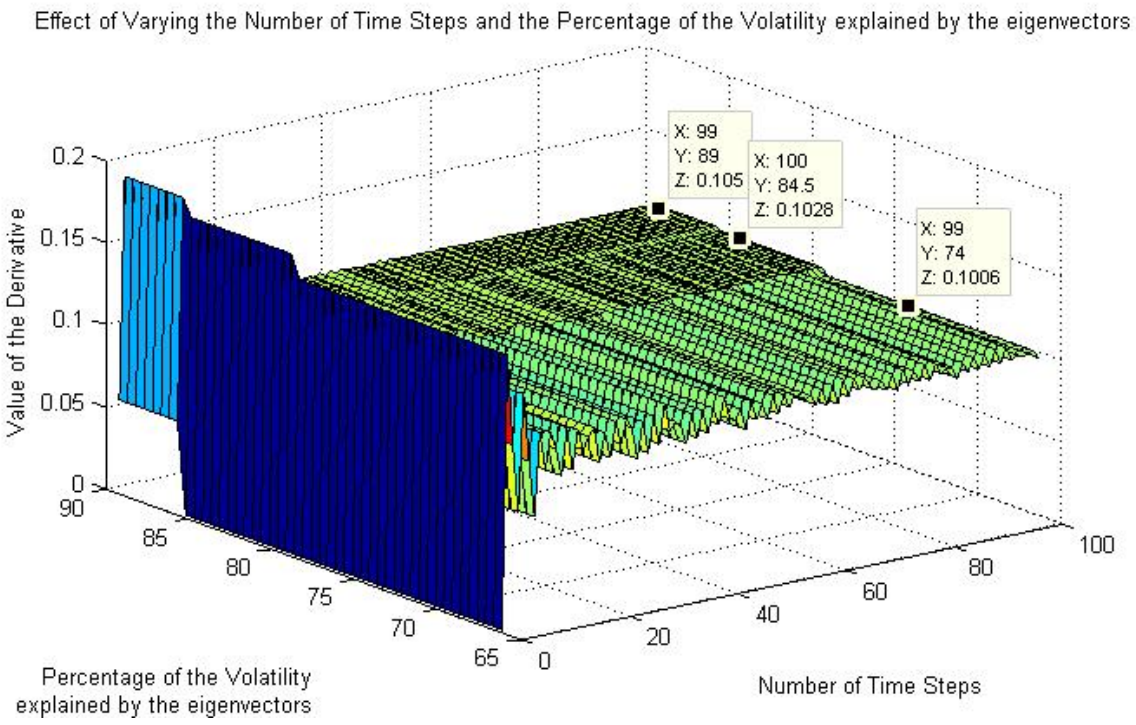

Figure 6: Effect of Varying the Number of Time Steps and the Percentage of variance explained by the Eigenvalues on Tranche 1

Remark 2. *To further clarify the usefulness of this methodology, we emphasize that it could be used for more than 2 eigenvalues if needed. The time-convergence seems reasonably fast on a standard computer as long as the number of eigenvalues is kept low (≤ 7, fortunately, as many as observed in real life data).*

6 Summary and Conclusion

This paper introduces several contributions to the area of multidimensional modelling in finance. Firstly, dimensionality and complexity have been reduced by proposing a method called stochastic principal components, which combines principal components analysis with stochastic dependence structure. The most important features implied by the model are stochastic volatility and correlation. It is shown that few eigenvectors and eigenvalues of the underlyings are sufficient to explain the joint behavior of the actual stocks/assets values. Secondly, we developed and implemented a tree model to price tranches of a CDO on a portfolio with stochastically correlated underlyings. This is achieved by means of building trees for the principal components and the stochastic eigenvalues with the purpose of computing rapidly the underlying probabilities of default in the portfolio of losses. This methodology relaxes the constant correlation assumption in the existing literature and also provides a fast reliable way of computing high dimensional derivatives. Furthermore, we provide a method to estimate the parameters of the stochastic processes of the principal components as well as those of the eigenvalues from real market data, this method is a generalization of the methods of moments for a large dimensional setting. For completion, the parameter estimation, the valuation of the tranches of a CDO and the speed of convergence of the method are provided as an example.

A Results of the Pricing of an Example. Mean Values

Here, the mean value using both path and non-path dependent methods for higher volatilities are presented. Tables 7 and 8 show a lower level of volatilities than tables 9 and 10.

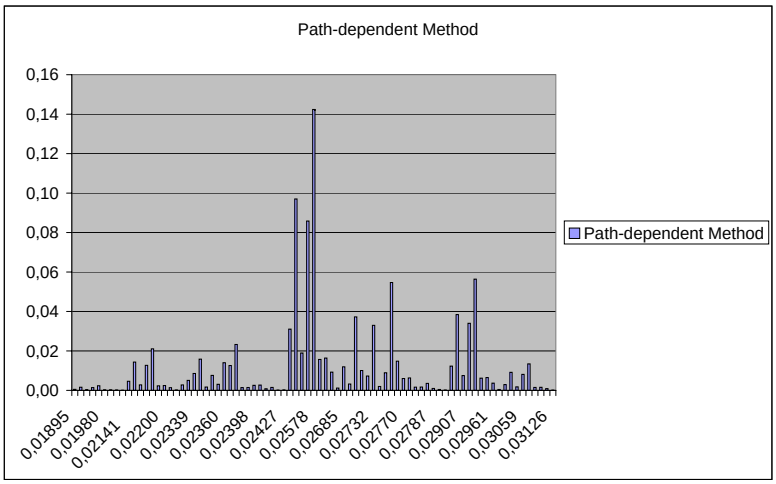

Figure 7: Mean Values using the path-dependent method

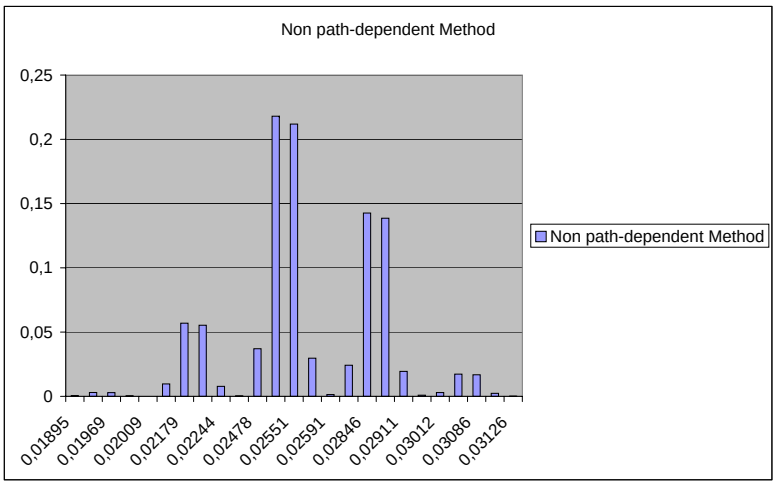

Figure 8: Mean Values using the non path-dependent method

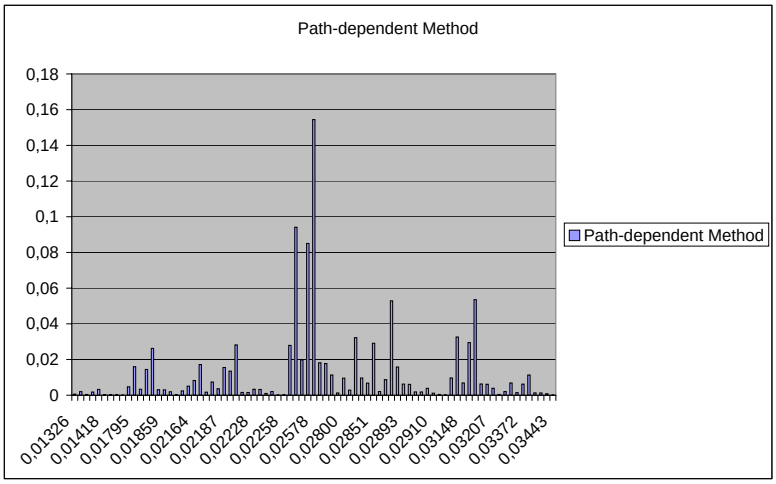

Figure 9: Mean Values using the path-dependent method

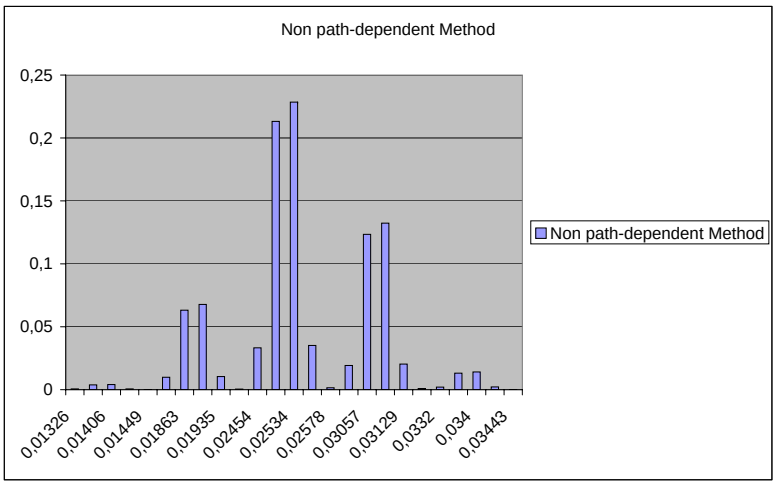

Figure 10: Mean Values using the non path-dependent method

References

[1] Alexander, C. (ed.) (2001). Orthogonal GARCH, In: *Mastering Risk Volume 2*, New York: Prentice Hall, 21-38.

[2] Alexander, C. (2000). A primer on the orthogonal GARCH model. *Manuscript.*

[3] Brennan, M.J. and E.S. Schwartz (1978). Finite difference method and jump processes arising in the pricing of contingent claims. *Journal of Financial Quantitative Analysis*, 13, 461-474.

[4] British Banking Association (2006). *Credit Derivatives Report 2006.*

[5] Broto, C. Ruiz E. (2004). Estimation Methods for Stochastic Volatility Models: A Survey. *Journal of Economic Surveys*, 18 (5), 613-649.

[6] Bjork, T. 1998 Arbitrage theory in continuous time. Oxford University Press.

[7] Cox, J.C. et al. (1979). Option Pricing: A Simplified Approach. *Journal of Financial Economics*, 7, 229-263.

[8] Duffie, D. and N. Garleanu (2001). Risk and Valuation of Collateralized Debt Obligations. *Manuscript.*

[9] Engle, R.F (2002) 'Dynamic Conditional Correlation - A Simple Class of Multivariate GARCH Models' Journal of Business and Economic Satistics, 20:3, 339-350.

[10] Escobar, M. et al. (2007). Pricing of Spread Options on stochastically correlated underlyings. *Manuscript.*

[11] Genon-Catalot, V. et al. (2000). Stochastic volatility Models as Hidden Markov Models and Statistical Applications. *Bernoulli*, 6 (6), 1051-1079.

[12] Hull, J. and A. White (1990). Valuing derivative securities using the explicit finite difference method. *Journal of Financial and Quantitative Analysis*, 25 (1), 87-100.

[13] Hull J., Predescu M., and White, A. (2005). The valuation of correlation-dependent derivatives using a structural model. *Manuscript.*

[14] Hull, J., and White, A. (2004). Valuation of a CDO and nth to Default CDS Without Monte Carlo Simulation. *Journal of Derivatives*, 12 (2), 8-23.

[15] Hurd, T. and A. Kuznetsov (2005). Fast CDO pricing in Affine Markov Chain Model of Credit Risk. *Manuscript.*

[16] Joliffe, I. T. (2002) Principal Component Analysis. 2nd ed. Springer.

[17] Laurent, J.-P. and J. Gregory (2003). Basket Default Swaps, CDO's and Factor Copulas. *Manuscript.*

[18] Lehmann, E. L. (2006) Nonparametrics. Statistical Methods based on Ranks. Springer, Revised First Edition.

[19] Merton, R.C. (1974). On the pricing of corporate debt: the risk structure of interest rates. *Journal of Finance*, 29, 449-470.

[20] Sørensen, H. (2000). Inference for Diffusion Processes and Stochastic Volatility Models. *PhD thesis.*

[21] Zagst, R. (2002). *Interest rate management*. Berlin: Springer Finance.