Abstract

We study some elements of pricing and risk management under multivariate models considering stochastic covariance. In particular, we price Mountain Range Derivative, Spread and Quantos under some of these models. Models based on Principal Components and Factor Analysis are considered in order to reduce dimension.

Keywords: stochastic correlation, principal components, pricing, risk management, factor model.

1. Introduction

Continuous time Stochastic Volatility models have been used in the pricing of financial derivatives, see Heston (1993). Natural extensions to a multivariate framework are models with stochastic covariance and correlations. They have shown to be useful in the pricing of financial derivatives involving several assets, also in has been on credit derivatives as for example Collateralized Debt obligations and k-th to default Credit Swaps (CDSs) among others (see Escobar et al (2008)).

Most models used in the pricing of multidimensional derivatives consider constant correlation among their constituents. Nevertheless, empirical facts suggest that correlation varies over time (see Engle (2002)). Neglecting changes in the correlation may introduce significant misleading in the pricing. In this paper we assess the impact of these models in bivariate equity derivatives like Spread Options, Quantos, Basket Options and Mountain Range Derivatives (MR). In particular we study Altiplano and Everest MR whose payoffs depend on a large number of underlyings.

One of the problems in dealing with such large dimensions is that they become rapidly intractable as the number of parameters growths exponentially. Some works introducing stochastic correlation in the pricing of financial derivatives are Van Emmerich (2006) and Escobar et al. (2007b).

In Van Emmerich (2006) the author provides analytical properties of the correlation process as well as an approximation to the value of Quantos options under the assumption of constant volatility for the underlyings.
In this paper, we introduce the multivariate Principal Component Stochastic Volatility Model (PCSV) as a simplified stochastic volatility model for asset prices.

This model has several theoretical and practical strengths. It tackles the curse of dimensionality by working in a relatively low dimensional space. It is critical when dealing with multidimensional derivatives as MR's which involve hundreds of underlying.

As a matter of fact, Principal Component Analysis, and the more general Factor Analysis, have been used to reduce dimension in the context of static covariance modelling. These studies are in support of the fact that few eigenvalues are sufficient to describe most of the variation in standard portfolios.

The organization of the paper is as follows:

In section 2 we introduce the PCSV model and its use in pricing Mountain Range Derivatives. In section 3 we study the risk management problem, in particular Value at Risk computation, under a Factor Stochastic Volatility model. In section 4 we provide approximated closed-form methods to price Spread options and other bivariate derivatives when assets have stochastic correlation. In section 5 we conclude.

2. Pricing Mountain Range Derivatives under Principal Component Stochastic Volatility Models

We define a PCSV model as:

**Definition** A $m$-dimensional stochastic process $Y(t)$ defined on a stochastic basis $(\Omega, \mathcal{A}, P, \mathbb{F}_t; t \geq 0)$ follows a PCSV $(m, p)$ model if it satisfies the following diffusion system:

$$
\begin{align*}
\frac{dY_j(t)}{dt} &= \left[ \mu_j - \frac{1}{2} \sum_{i=1}^{p} a_{ij}^2 V_j(t) \right] dt + \sum_{i=1}^{p} a_{ij} \sigma_j(t) dW_j(t) \\
\frac{dV_j(t)}{dt} &= \gamma_j (\beta_j - V_j(t)) dt + c_j \sqrt{V_j(t)} dB_j(t)
\end{align*}
$$

We assume $\gamma_j > 0$, $\beta_j \gamma_j \geq \frac{\epsilon^2}{2}$ for $j = 1, 2, \ldots, p$ and, without lost of generality, $0 < \beta_j \leq \beta_{j-1}$.

Here:

1) $< W_j(t), B_j(t) > = \rho_{ij}$ for $j = 1, 2, \ldots, p$ where, is the standard quadratic variation of two processes.

2) $V_j(t) = \sigma_j^2(t)$ with initial conditions $V_j(0) = \eta_j$, where $\eta_j$ is a random variable and $Y(0) = y_0$ for some real value $y_0$.

3) Moreover $\sigma_j(t)$ and $a_j = (a_{ij}, \ldots, a_{ip})$ represent the eigenvalues at time $t$ and the eigenvectors of the instantaneous covariance matrix $\Sigma(t, dt)$ of the underlying process $\exp(Y)$, i.e. the asset prices. $A = (a_y)$ denotes the $m \times p$ dimensional matrix of the eigenvalues.

Stochastic correlations under a SVPC model, estimation by the Method of Moments and its asymptotic properties, such as stationarity and mixing, have been studied in Escobar et al.(2009).

For any fixed $t>0$ the process decomposes into $p$ orthogonal directions given by eigenvectors $a_i$ non depending on $t$. Moreover, the number of parameters in the PCSV model is $p(m+2) + m$, which can be fairly small compared with number of parameters when considering the complete covariance matrix of order $m$. For example if we consider $m = 100$ and $p = 5$ it will give 610 parameters instead of the usual 5050 in a full stochastic covariance matrix.

Another advantage of the PCSV model is that the conditional characteristic function can be obtained in closed form. As the eigenvalue processes are not observable, a Hidden Markov Chain approach is required for estimation purposes (see Escobar et al (2009)).

A natural question coming out is how many principal components are needed to explain the volatility? We apply a principal component technique to a series of daily returns of S&P 100 from 1983 to 2006. As can
Figure 1. Number of Principal Components versus Cumulative Variance Explained. A large percentage of the total variability is explained with a few principal components.

Figure 2. Estimation by the Method of Moments in a PCSV models with three components from simulated data. Global error decreases slowly with the sample size.

be seen in Figure 1, the total expected variance explained grows with the number of components. Moreover, less than seven components explain 97% of the variability. Figure 2 shows the error in the parameter estimation (lower curve) and the estimation of the eigenvalue matrix (upper curve).

As we can see the global error decreases slowly with the number of observations, consistently with the asymptotic properties of the Method of Moments estimator. All parameters are within the corresponding 95% confidence intervals.

2.1. Remarks
1. These estimators may be used eventually as the initial point for other more efficient estimation methods.
2. In correspondence with the lack of efficiency of the Method of Moments, the asymptotic variances are far from being optimal, or equivalently, a relatively large series of returns in needed to obtain accurate estimators.
MR are exotic financial options originally marketed by Société Générale in 1998. These options combine the characteristics of Basket Options and Range Options by basing their payoff on several underlying assets. If \((S_1,\ldots,S_n)\) is a vector of underlying stocks we have the following payoffs:

1. Altiplano (simple version): \(\min_{\{i\in\mathbb{Z}\}} S_i(t) > D\)

2. Everest: \(\min_i S_i(T)\).

We study the sensitivity of their prices to the changes in the parameters of the PCSV model through simulated data coming from a SVPC model. We assume the following parametric setting for the simulations in this subsection. The maturity time is 4 years, the interest rate is \(r = 0.1\), the number of companies is 50, maturity time is set at \(T = 100\) days and the number of eigenvalues is \(p=3\). The barrier is \(D = 0.95\). The set of parameters are \(\beta_{1,2,3} = 0.8, 0.72, 0.64\), \(c_{1,2,3} = 0.4, 0.36, 0.32\), \(\rho_{1,2,3} = 0.4\).

In Figures 3 and 4 some prices of Altiplano versus mean reverting level, volatility and correlation of the eigenvalues are shown.

![Figure 3. Sensitivity of Altiplano prices with respect to the mean reverting level.](image1)

![Figure 4. Sensitivity of Altiplano prices with respect to the mean reverting level.](image2)
2.2. Remark
Simulated prices show the following facts:
1. Larger \( \beta \) s imply higher volatility of stocks, which implies in turn higher probabilities to be below the barrier at maturity.
2. Larger values of \( \rho \) imply lower volatilities of stocks and hence lower probability to be below the barrier at maturity.
3. Large values of \( c \) imply high volatility of the eigenvalue process therefore lower probability to be below the barrier at maturity. (large eigenvalues have less impact than small ones).

3. Computing Value at Risk under Factor Stochastic Volatility Models
We extend the Principal Component Stochastic Volatility Model by considering an additional factor in the drift of each asset, while keeping the dynamic in the eigenvalues of the covariance matrix. The use of common factors explaining the dynamic of the assets and their volatilities is a quite standard technique in finance and statistics. A Factor Stochastic Volatility Model (FSV) is defined as follow:

**Definition** A m-dimensional stochastic process \( Y(t) \) follows a FSV if it satisfies the following equation:

\[
\begin{align*}
    dY_i(t) &= \left( \mu_i - \frac{1}{2} \sum_{j=1}^{p} \sigma^2_{ij} \right) Y_i(t) dt + \frac{1}{2} \sum_{j=1}^{p} \sigma_{ij} (t) dW^{(i)}_j(t) \\
    dV_j(t) &= \gamma_j (\beta_j - V_j(t)) dt + c_j \sqrt{V_j(t)} dB_j(t) \\
    dV^{(i)}(t) &= \gamma^{(i)} (\beta^{(i)} - V^{(i)}(t)) dt + c^{(i)} \sqrt{V^{(i)}(t)} dB^{(i)}(t)
\end{align*}
\]

\( W_j(t), V^{(i)}(t), B^{(i)}(t) \) and \( B_j(t) \) are Brownian motions with quadratic variations:

\[
\begin{align*}
    <W_j(t), B_j(t)> &= \delta_{jj} \rho_t \\
    <W_j(t), V^{(i)}(t)> &= <B_j(t), V^{(i)}(t)> \\
    &= <B_j(t), B^{(i)}(t)> = 0
\end{align*}
\]

Here \( V_j(t), j = 1, 2, ..., p \) represent \( p \) common factors influencing the asset prices. Expressions for stochastic correlations between assets, factors and correlation are obtained. Closed form expressions for the conditional CF are available (see Escobar et al. (2008)).

The FSV is a simplified stochastic volatility model for the logarithm of stock prices. This model has several theoretical and practical strengths. First, it allows reducing the dimensionality. Secondly, for eigenvalues and volatilities of the intrinsic factors we suggest a system of stochastic processes capturing several "levels" of stochasticity: stochastic volatility for the underlying, stochastic correlation among stocks, variances and stocks-correlations.

This stochasticity has been argued to capture several stylized features of the historical data as well as of derivatives, for instance smile, skew volatilities and correlations, see for example Ball and Torous (2000).

We study the problem of parameter estimation for these classes of processes from observations at discrete times. In particular we use empirical moments and obtain confidence intervals based on their ergodic properties (see Escobar et al (2008)) whose results can be easily extended to our framework. Factor models are not easy to estimate because available methods for handling common factors are based on a recursive estimation of both factor loading and commonality, where iterations are performed until the commonality values converge (see Comrey and Lee (1992)).
It is known that, for large covariance matrices, few iterations bring the commonalities closed enough to convergent values. In our analysis we consider, in order to simplify, only one factor and three assets but the technique can be extended to a larger number of factors and assets in a straightforward way.

The study is based on three representative Asian funds: Korea Fund Inc., Pacific Fund Inc. and ROC Taiwan Fund.

Figure 5 shows the correlations among the three funds log-prices. The empirical correlation coefficient is computed for non overlapping quarters. As it can be observed, the empirical correlations do not appear to be constant in time. Moreover, they seem to follow a random pattern. After year 1997 we observe a change in the trend and even more erratic behavior that could be explained by the Asian crisis.

Factors are estimated by Principal Factor Method (see Comrey and Lee (1992)). After removal the factor, empirical estimators for the parameters of the Cox-Ingersoll-Ross process (CIR) are obtained following (see Overbeck and Rydén(1997)).

Asymptotic normality results are available, therefore confidence interval can be obtained. All parameters, except $\mu$, are relevant, particularly the mean reverting level and the volatility of the factor volatility given by: $(\beta_1, c_1, \beta^{(2)}, c^{(2)}, \beta^{(3)}, c^{(3)})$.

Next, we address the issue of the sensitivity of the parameters in the model with respect to the VaR. The notion of sensitivity to the VaR allows to gauge the statistical errors in the calibration of the parameters. It expresses how much small errors in the calibration procedure of the true value of the parameters translate into moderate to large changes in the VaR. We consider simulated data from an equally weighted portfolio composed of three assets whose prices are varying along under the factor model. The risk measure is computed at different levels of the set of parameters $(\beta, c)$ and finally the relative changes in the VaR with respect to the parameter $\beta$ are measured through the formula:
for small values of $h$. In Table 1 sensitivities are shown.

<table>
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<th>c\beta</th>
<th>0%</th>
<th>80%</th>
<th>111</th>
<th>240%</th>
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<td>-292.76</td>
<td>-335.77</td>
<td>-334.14</td>
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<tr>
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<td>-13.11</td>
<td>-25.02</td>
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<td>-8.31</td>
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<tr>
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<td>5.62</td>
<td>-2.41</td>
<td>-7.43</td>
<td>-13.38</td>
</tr>
</tbody>
</table>

They are obtained from simulated data in the one factor-three funds model. The parameters are measured on year percentages. We see how, in general, the VaR is quite sensitive to changes on the parameters, particularly when the parameters $\beta$ and increase the impact of small changes in the parameters lead to large changes in the VaR.

4. Pricing Bivariate Derivatives Under Stochastic Correlation Models

Examples of two-dimensional continuous time models that introduce constant correlation on leading Brownian motions have been used for modeling and pricing Spread Options, see Carmona (2003).

These models have the nice property that the law of the bi-dimensional random variable representing the value of the assets at maturity time has a tractable form. It implies that the pricing problem for European type options can be solved with relative ease, providing accurate closed-form approximations (see Deng(2008)) and even in some cases it is possible to provide exact closed forms expression, see for example Wilmott (2000).

In this section we give approximations to the pricing of bivariate option like Quantos and Spreads under models with stochastic correlation structure. For details and proofs see Alvarez et al (2009). We consider the following bivariate model:

$$
\begin{align*}
  dS_1(t) &= S_1(t)[\mu_1(t, S(t)) dt + \sigma_1(t, S_1) dW_1(t)] \\
  dS_2(t) &= S_2(t)[\mu_2(t, S(t)) dt + \sigma_2(t, S_2) dW_2(t)]
\end{align*}
$$

where

$$
W_1(t) = \int \rho(s, S_1) dW_1(s) + \int \sqrt{1 - \rho(s, S_1)^2} d\tilde{W}_2(s)
$$

and $W_1$ and $\tilde{W}_2$ are two independent Brownian motions.

$$
E[dW_1(t), dW_2(t)] = \rho(t, S_1) dt
$$

For the model above, the correlation, denoted to simplify by $\rho$, is given by:

$$
d\rho = a(m - \rho) ds + c\sqrt{1 - \rho^2} dB_s
$$

where $B$ is a Brownian motion independent of $W_1$ and $\tilde{W}_2$. The stochastic dynamic for $\rho$ that we consider in the examples are:

1. Mean reverting square root process, which was already studied in Von Emmerich (2006) is the process satisfying the SDE:

$$
d\rho = a(m - \rho) ds + c\sqrt{1 - \rho^2} dB_s
$$
where $B$ is a Brownian motion independent of $W_1$ and $W_2$.

Some properties of this stochastic process are obtained in Van Emmerich (2006).

2. The Markov Correlation Switching model. This model is a continuous time Markov Chain with state space $(\rho_l)_{l \in l}$, where $l$ is a finite set.

These models are flexible enough to represent a large variety of correlation behaviors.

Let $C_f$ be the class of European type derivatives whose payoff $h$ depends on the bivariate vector $S(T)$, where $T$ is the maturity time.

Our main objective is to price derivatives on $C_f$ under the assumption of stochastic correlation. In particular we analyze, two cases of derivatives in $C_f$:

1. Spread Options: with payoff: $h(S_1(T),S_2(T)) = (S_1(T) - S_2(T) - K)^+$
2. Quantos Options: with payoff: $h(S_1(T),S_2(T)) = (S_1(T)S_2(T) - K)^+$

As usually $a^* = \max(0,a)$. First, note that under deterministic $\rho$, the vector $l^*$ has a bivariate normal distribution of mean

$$\left( \mu_1 - \frac{\sigma_1^2}{2} T \right)$$

and covariance matrix

$$\begin{pmatrix}
\sigma_1^2 T & \sigma_1 \sigma_2 \int_{0}^{T} \rho_s \, ds \\
\sigma_1 \sigma_2 \int_{0}^{T} \rho_s \, ds & \sigma_2^2 T
\end{pmatrix}$$

One of the consequences of the previous result is that the probability distribution of $l^*$ in the case of $\rho$, deterministic is the same probability distribution than the one assuming constant correlation equal to the mean value of the correlation over the interval $[0,T]$

$$\rho^*(T) = \frac{1}{T} \int_{0}^{T} \rho(s) \, ds.$$ 

Denote by $\Pi_0(\rho)$, the price at time $t=0$ of a derivative in $C_f$ under a model with constant coefficient $\rho$, hence under deterministic correlation coefficient $\Pi_0(\rho^*(T))$.

In the stochastic correlation framework the price is found as:

$$p = e^{-rT} E_0 h(S_1(T),S_2(T))$$

$$= E_0 \left( e^{-rT} E_0 (h(S_1(T),S_2(T)) \mid \rho^*_T) \right)$$

$$= E_0 \Pi_0(\rho^*_T)$$

It is key for pricing derivatives in the context of stochastic correlation but the usefulness could be limited when the probability law of $\rho^*_T$ is unknown. Therefore we provide a methodology to obtain an expression for $E_0 \Pi_0(\rho^*_T)$ as well as bounds for the approximating error under very general conditions.

### 4.1. First order approximation

Suppose that $\hat{\Pi}$ is the Taylor linear approximation of $\Pi$ around the point $r^* = E_0(\rho^*_T)$, i.e.

$$\hat{\Pi}(\rho) = \Pi(r^*) + \Pi'(r^*)(\rho - r^*)$$

then the first order approximated price $p_1$ satisfies:

$$p_1 = E_0 \hat{\Pi}(\rho^*_T) = \hat{\Pi}(E_0(\rho^*_T)) = \hat{\Pi}(r^*) = \Pi(r^*) = E_0(\rho^*_T)$$

To estimate the approximation error we use the remainder of Taylor's formula. We know that there exists $\tilde{\rho}$.
\[ \Pi(\rho) = \Pi(r^*) + \Pi'(r^*)(\rho - r^*) + \frac{1}{2} \Pi''(\rho)(\rho - r^*)^2 \]

The first two terms of this sum corresponds to \( \tilde{\Pi}(\rho) \) so
\[ \Pi(\rho) - \tilde{\Pi}(\rho) = \frac{1}{2} \Pi''(\rho)(\rho - r^*)^2 \]

Evaluating in the random variable \( \rho^*_r \) and taking expectations in both members with respect to \( Q \) we have
\[ E_\rho \Pi(\rho^*_r) - E_\rho \tilde{\Pi}(\rho^*_r) = \frac{1}{2} E_\rho \Pi''(\rho^*_r)(\rho^*_r - r^*)^2 \]
\[ |p - p_1| = \left| \frac{1}{2} E_\rho \Pi''(\rho^*_r)(\rho^*_r - r^*)^2 \right| \leq \frac{1}{2} E_\rho \left| \Pi''(\rho^*_r) \right| (\rho^*_r - r^*)^2 \leq \frac{1}{2} \sup_{-1 < \rho < 1} \left| \Pi''(\rho) \right| E_\rho (\rho^*_r - r^*)^2 = \frac{1}{2} Var_\rho (\rho^*_r) \sup_{-1 < \rho < 1} \left| \Pi''(\rho) \right| \]

4.2. Second order approximation

The second order approximation to the derivative price is obtained using the second degree Taylor's polynomial \( \tilde{\Pi} \) around \( r^* \):
\[ \tilde{\Pi}(\rho) = \Pi(r^*) + \Pi'(r^*)(\rho - r^*) + \frac{1}{2} \Pi''(r^*)(\rho - r^*)^2 \]

In this case the second order approximated price is
\[ p_2 = E_\rho \tilde{\Pi}(\rho^*_r) = \Pi(E_\rho (\rho^*_r)) + \frac{1}{2} \Pi''(E_\rho (\rho^*_r)) Var_\rho (\rho^*_r) \]

Analogously, a bound for the error of \( p_2 \) can be obtained using Taylor's formula in terms of \( m_3(\rho^*_r) \) the third centered absolute moment of \( \rho^*_r \), and the third derivative of \( \Pi \):
\[ |p - p_2| \leq \frac{1}{6} m_3(\rho^*_r) \sup_{-1 < \rho < 1} \left| \Pi'''(\rho) \right| \]

Under the bivariate model, given the value of \( \rho_o \), the expectation and the variance of random variable \( \rho^*_r \) are given by:
\[ E(\rho^*_r) = m + (\rho_o - m) \frac{1 - e^{-at}}{at} \]

4.3. Remarks
1. Ito's formula provides a closed expression for \( Var(\rho^*_r) \).
2. For a switching model formulas for \( E(\rho^*_r) \) and \( Var(\rho^*_r) \) can be obtained.
3. Quantos and Product Options can be treated in a similar way.
4. For some set of parameters the price is linear on \( \rho \).
5. For large maturity, due to the ergodicity of the process $\rho^\star_t$ converges to a deterministic and known value. The convergence is of the order of $\sqrt{t}$.

In Figure 6 we show the almost linear behavior of the constant correlation price function $\Pi(\rho)$. We present the actual price computed by MC simulations and the linear and quadratic Taylor's approximations for a negative, zero and a positive correlation. Approximation is better in the quadratic approximation than in the linear one, as expected.

![Figure 6](image.png)

*Figure 6.* Spread Option price as function of the correlation coefficient together with linear and quadratic approximations.

Simulation studies in Alvarez et al (2009) suggest a considerable gain in computation time compared with Monte Carlo and partial Monte Carlo (i.e. only simulating the correlation).

5. Conclusions and Recommendations

We have shown the impact of stochastic correlation in pricing and risk measures. As a consequence we demonstrate that neglecting this empirical fact may result in an underestimation of risk or a mispricing of multivariate financial derivatives.

On the other hand introducing stochastic correlation in the model creates additional complexities when it comes to estimation and finding expected values of some functionals of the process. We propose some settings where it is possible to obtain a closed-form for its characteristic function, therefore to compute moments and expected values.

The study of multivariate models with stochastic correlation and their financial applications is still in an early stage. In particular there is a strong need of efficient method for calibration. Their impact in pricing CDOs remains still a challenge, but simple models like SVPC and FSV may open a path toward more realistic results outside the Gaussian framework.
6. Bibliography


